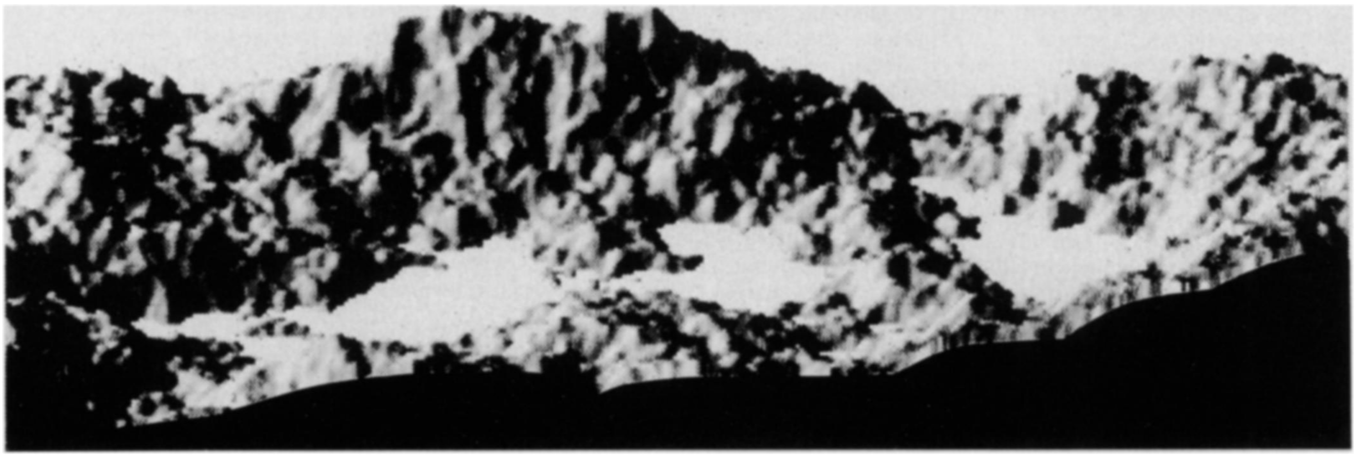
FRACTALS: A World of Nonintegral Dimensions



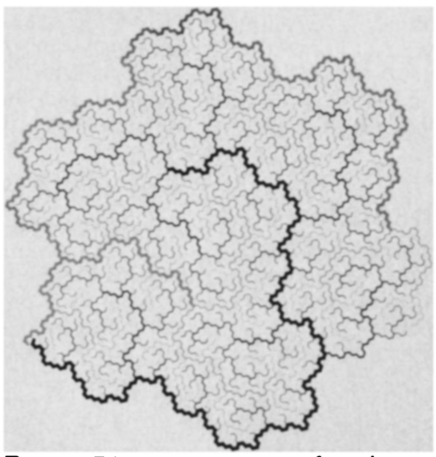
Brownian Landscape: *A computer-based mountain scene based on controlled Brownian motion bears striking resemblance to actual scenes (especially of lunar landscapes). The dimension of this surface is $2\frac{1}{4}$. Similar models based on a target dimension of $2\frac{1}{2}$ prove too erratic when compared to reality.*

Science routinely involves entities that are in 1, 2, 3 and 4 dimensions. Now, there are new insights into well-known but enigmatic phenomena whose dimensions are other than whole numbers.

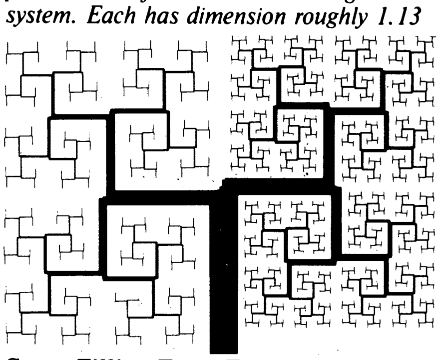
BY LYNN ARTHUR STEEN

The world of classical geometry is inhabited by objects of integer dimension: Spheres, cubes and other solids are three-dimensional; squares, triangles and other plane figures are two dimensional; lines and curves are one-dimensional, and points are zero-dimensional. The basic categories in which size is measured—volume, area, length—reflect this fundamental classification, as do virtually all theories of physics and physical chemistry. Even when more sophisticated theories (e.g., relativity, quantum mechanics, linear algebra, Hilbert space) have required more than three dimensions, the extension to higher dimensions progresses in integer steps (until it becomes infinite). Dimension, if finite, has always been measured in integers.

Nevertheless, for the last hundred years there have been infrequent but reliable reports of both mathematical and scientific phenomena that behave as if they had a dimension part way between two whole numbers. Now these scattered reports have been unified into a new theory of *fractals*, objects that do not conform to classical definitions of dimension. Typically, fractals are extremely irregular curves or surfaces that wiggle enough to partially fill the gap between one dimension and the next higher one.



Fractal River: *An intricate fractal computer model of a river and drainage divide system. Each has dimension roughly 1.13*



Space Filling Tree: *The growth of a fractal tree provides an idealized model for root or arterial systems, or for the structure of the lung. The dimension of this tree is between 1 and 2.*

One of the oldest (and still one of the most striking) examples of a fractal in geometry is the "snowflake" curve introduced by the German mathematician Helge von Koch in 1904. Construction of this curve begins with an equilateral triangle. The middle third of each side of this triangle is stretched outwards until it forms two sides of a smaller equilateral triangle; the resulting figure, stage two in the construction of the Koch curve, is a Star of David. Stage three is formed by stretching outward similarly the middle thirds of each of the 12 sides of the Star of David. This process is repeated *ad infinitum*, producing as a limit the Koch curve.

This curve has certain very remarkable properties. At each stage in its construction, the length of the curve increases by four-thirds, yet the area enclosed increases by only one-third. Hence, the final curve has infinite length but encloses a finite area. The Koch curve is continuous yet has sharp corners almost everywhere. This means that, for the most part, it does not have tangents and, technically speaking, is not differentiable. Because of this bizarre behavior, the Koch curve was termed "monstrous" by many of Koch's contemporaries and was widely believed to be a mathematical pathology totally unrelated to any possible real world phenomenon.

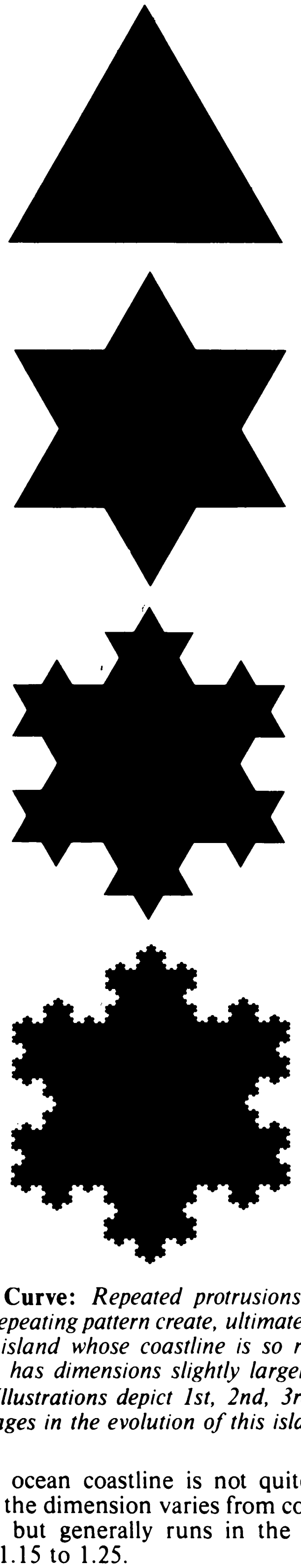
The Koch curve does, however, bear a striking resemblance to the shape of a rugged coastline. Indeed, in comparison to a coastline, the Koch curve is excessively regular: its peninsulas and bays occur with absolute precision. Yet both the Koch curve and typical coastlines have a fundamental property that is at the root of all fractal phenomena: the general nature of their shape—the extent of bends and wiggles, not the precise shape or location of them—looks the same no matter what scale is used for measurement.

If you look at a map of a rugged coastline drawn at a scale of 1:100,000 and then again at a scale of 1:10,000 innumerable sub-bays and peninsulas become visible; the same thing happens at a scale of 1:1,000 and so forth *ad infinitum.* Analogously, but more regularly, changing scales by a factor of three in a drawing of the Koch curve reveals new patterns of triangular bumps that in each case look similar to the previous sketch. This feature of similar patterns revealed under change of scale is called *self-similarity* and is at the heart of most fractal objects.

One consequence of self-similarity (for coastlines and for some other phenomena as well) is that length no longer provides an adequate measure of size, for if a coastline is measured with shorter and shorter measuring sticks, its length grows without bound. In fact, every one of a half-dozen reasonable definitions of the length of a coast leads to the conclusion that the true length is infinite—because the extent of wiggling is too great. Even though a coastline, being a curve, is geometrically one-dimensional, the method of measurement appropriate to one-dimensional objects is ineffective.

The problem can be resolved, according to mathematician Benoit Mandelbrot an IBM Fellow at IBM's Thomas J. Watson Research Center in Yorktown Heights, N.Y., by recognizing that the dimension of an object must be used as an exponent in measuring its size. From this point of view the actual dimension of a coastline (or of the Koch curve) is *not* one. If a line segment is divided into N similar parts, each part reduced from the original by a scale factor r, then $N = 1/r$ and the total length of the original segment equals $1/r$ times the length of each part. Likewise, if a square is divided into N similar parts, each reduced from the original by a scale factor r, then $N = 1/r^2$, so the total area of the original square equals $1/r^2$ times the area of each smaller square. In other words, when a geometric figure is rescaled, its dimension (1 for the line, 2 for the square) must be used as an exponent in reconstituting the measure of the whole as the sum of the measure of its parts. The general formula is $N = 1/r^d$, where N is the number of similar parts scaled by the factor r, and d is the appropriate dimension. This fact provides a quantitative tool for estimating dimensions of fractals.

For example, if a segment of the Koch curve is divided into $N = 4$ parts, each part will be similar to the original but reduced by the fraction $r = 1/3$. Hence, if the pattern $N = 1/r^d$ is to hold, then d must equal $\log N/\log (1/r)$; in this case, $d = \log 4/\log 3 = 1.2618$. So the fractional dimension of the fractal Koch curve is 1.2618. Empirical studies by the eccentric British meteorologist Lewis Fry Richardson show that the dimension of



Illustrations: *Fractals: Form, Chance, and Dimension* by Benoit B. Mandelbrot, W. H. Freeman and Co., © 1977 by author.

Koch Curve: *Repeated protrusions in a strict repeating pattern create, ultimately, the Koch island whose coastline is so rugged that it has dimensions slightly larger than $1\frac{1}{4}$. Illustrations depict 1st, 2nd, 3rd and 6th stages in the evolution of this island.*

actual ocean coastline is not quite this large; the dimension varies from coast to coast, but generally runs in the range from 1.15 to 1.25.

Fractals arise in many parts of the scientific and mathematical world. Sets and curves with the discordant dimensional behavior of fractals were introduced at the end of the 19th century by Georg Cantor and Karl Weierstrass. Until now their use has been limited primarily to theoretical investigations in advanced mathematical analysis. Like the Koch curve, they were considered too bizarre for application to the real world.

The primary scientific example of fractals has been Brownian motion: It exhibits the excessive irregularity and statistical self-similarity typical of fractal phenomena. When a Brownian path is examined in increasing detail, its length—like coastlines—increases without bound. Jean Perrin, whose work on Brownian motion won him the Nobel Prize in physics, observed the similarity between Brownian paths and the non-differentiable "monster" curves of the mathematicians and attributed it to the self-similarity phenomenon that Mandelbrot has now identified as the source of fractal behavior: "[A]ny scale," said Perrin, "involves details that absolutely prohibits the fixing of a tangent .... An unprejudiced observer would therefore conclude that he is dealing with a function without derivatives ...."

Mandelbrot, in a recently published book entitled *Fractals: Form, Chance and Dimension* (Freeman, 1977, $14.95), shows how these classic examples of fractals provide insight into a host of scientific observations previously lacking a unified theory. For example, from the fractal theory of coastlines, Mandelbrot explains the observed relation between the number and size of islands in an archipelago, as well as between the number and size of lakes (the opposite of islands) in a continent.

Similar analysis explains the so-called Zipf phenomenon in statistical linguistics. The statistical distribution of word frequencies in different languages is nearly universal and follows an empirical curve that depends on a certain exponent. Mandelbrot derives the Zipf law from the self-similar characteristic of a lexicographical tree, and shows how dimension, in this context, is a measure of the richness of the vocabulary. A similar analysis leads to the empirically based Pareto law for salaries.

In 1890, Giuseppi Peano created a fractal "space-filling" curve. It meandered so much that it passed through every point in a unit square! This behavior, suitably modified, is a good idealization of the geometry of river networks and of the vascular network in a body: To do their job, rivers and veins must pass within a small distance of every point of the territory they serve. Hence, these networks, too, are examples of fractal phenomena.

Other examples brough under the fractal umbrella by Mandelbrot's recent work include the clustering and distribution of stellar matter, cratering of the moon, geometry of polymers, turbulence in fluids and the distribution of errors in data transmission. In each case, the analysis depends on the use of well-established mathematical curves or surfaces "thus far reputed pathological." Fractal behavior erupts whenever self-similarity forces the whole to be, in certain essential respects, the same as its parts. Richardson expressed this well when he wrote of turbulence:

Big whorls have little whorls
which feed on their velocity;
And little whorls have lesser whorls,
and so on to viscosity. □