

WHAT'S IN A GAME?

Mathematical games offer more than fun (and frustration)—they can provide insight into mathematical theory

BY LYNN ARTHUR STEEN

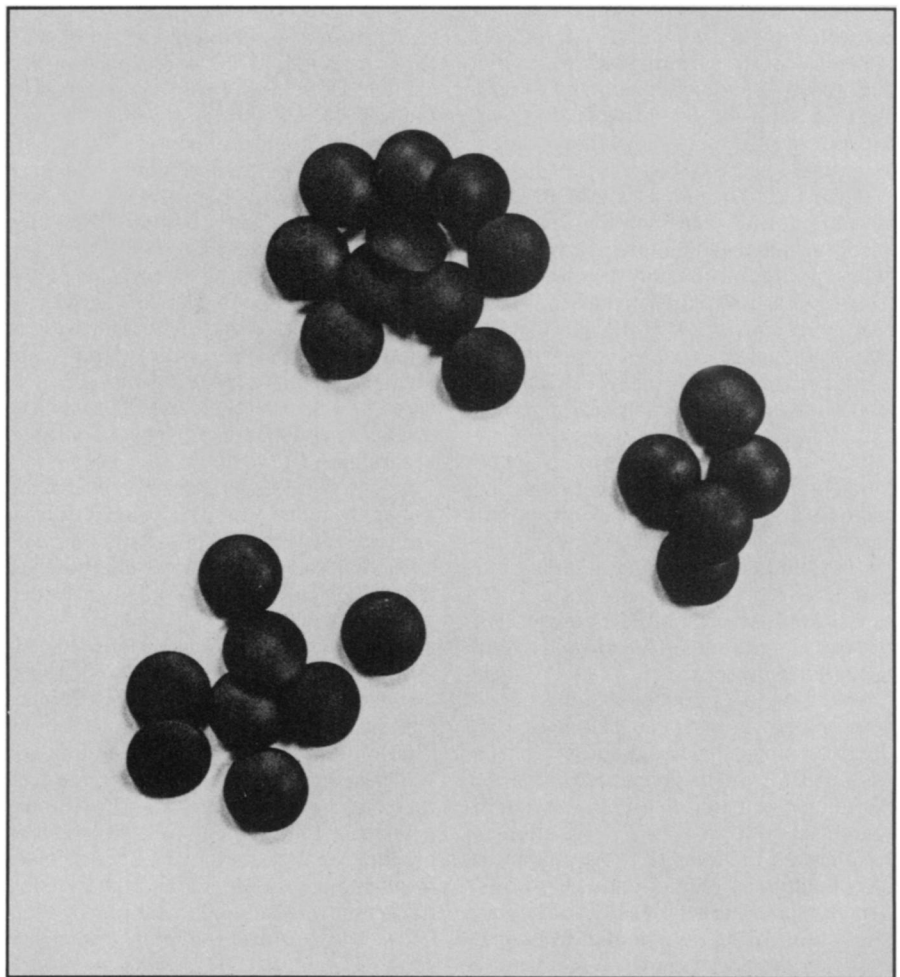
Take three piles of stones, with perhaps 5 to 10 stones in each pile. Two players take turns removing stones, at each turn removing one or more stones from any *single* pile. The player who removes the last stone is the winner.

This game, a bit more subtle than tic-tac-toe, was created in 1902 by the Harvard University Mathematician Charles Bouton. He called it "Nim," perhaps after the German *nimm*, meaning "take." It has had a limited yet enthusiastic following among mathematicians and engineers because there is a perfect strategy for playing the game that involves binary numbers. This has also made it a popular computer game, since the winning strategy can be easily programmed.

Nim is just one example of a growing interaction between mathematical theories and games of strategy. Like games, mathematical theories are regulated by precise rules, some of which may be altered or abolished in order to increase interest, variety or elegance. Both the mathematician and the game player rely on extensive hypothetical ("what if...") reasoning to develop sound strategy. So it is not surprising that games interact with mathematics, each providing insight into the process of the other.

The connection between game strategies and mathematical theory is best illustrated by the reasonably well-known system of binary numbers. These numbers use the digits 0 and 1 to represent powers of two (1, 2, 4, 8, ...), just as decimal numbers use the digits 0, 1, 2, ... 9 to represent powers of ten (1, 10, 100, 1,000, ...). A pile of, say, 5 stones in a Nim game is represented in binary notation by the number 101 because 101 means $1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$, which is just $4 + 1 = 5$. If the other two piles contain $6 = 4 + 2$ (in binary, 110) and $7 = 4 + 2 + 1$ (in binary, 111) stones, then the Nim position of 5, 6 and 7 stones is represented in binary form by the code numbers 101, 110, and 111. These three numbers can be summarized by adding them in a special way, called Nim-addition: Use ordinary addition with the special rule that $1 + 1 = 0$. Thus

$$\begin{array}{r} 101 \\ + 110 \\ + 111 \\ \hline 100 \end{array}$$



Appleman

Hence, the Nim position 5, 6, 7 is summarized by the binary number 100.

The connection between binary numbers and Nim strategy is this: Those positions that are "safe" for players to leave when they complete a move are the ones that have a binary Nim total of 0. Leaving no stones is the safest of all, because the player who does that wins. Leaving some other collection of stones whose Nim total is 0 is equally safe, for your opponent then cannot take enough stones to also leave a total of 0. (This is not exactly obvious, but can be easily verified by playing a few games.) So in your next turn you can take

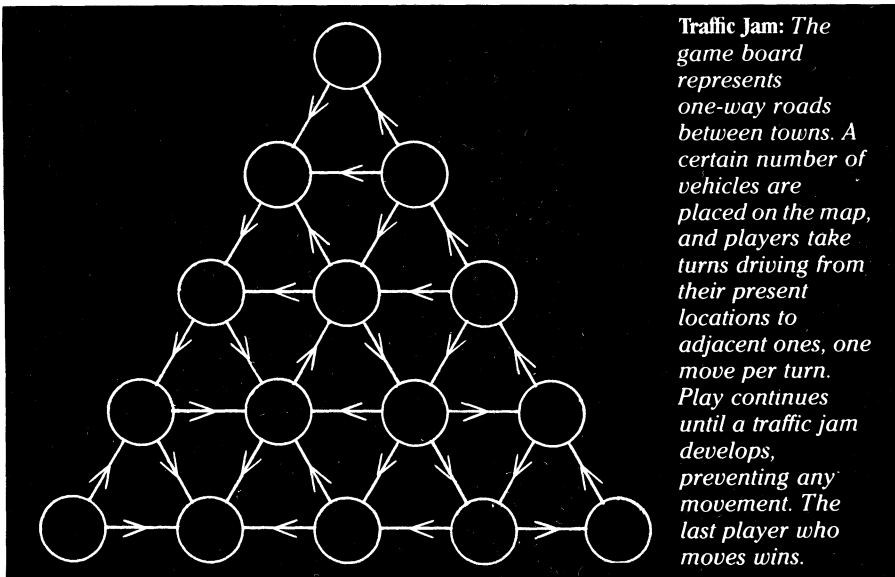
away as many stones as the Nim total your opponent left, thus restoring the game to a safe position (for you) with Nim total of 0.

For example, faced with piles of size 5, 6, and 7, whose Nim total is 100 (in binary) or $2^2 = 4$ (in decimal), you would take away 4 stones from any pile. This would leave piles either of sizes 1, 6, 7, or 5, 2, 7, or 5, 6, 3—and each of these has a Nim sum of 0. No matter what your opponent does next, you can continue this tactic until you reduce the piles to 0—and win.

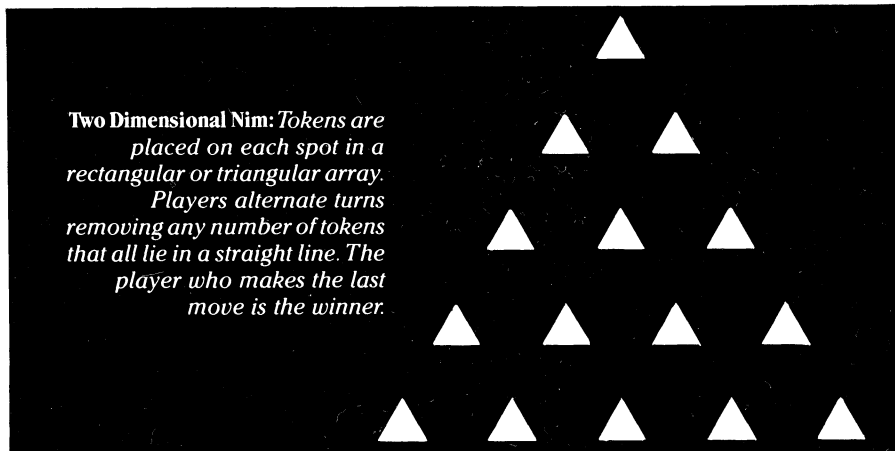
Nim addition, first introduced by Bouton in his analysis of the game, is a useful but strange alternative number sys-

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Welter: A number of tokens are placed at different locations on a strip of paper marked into squares. Each player in turn moves a single token to any unoccupied square to the left, jumping over other tokens if desired. The player who makes the last move wins. Two variations of the rules are common. In one, tokens may be placed only in any unoccupied square to the left that can be reached without jumping over other tokens. In the other, tokens may move only one square at a time, but may be moved to an occupied square. In this case, both tokens are removed from the board.

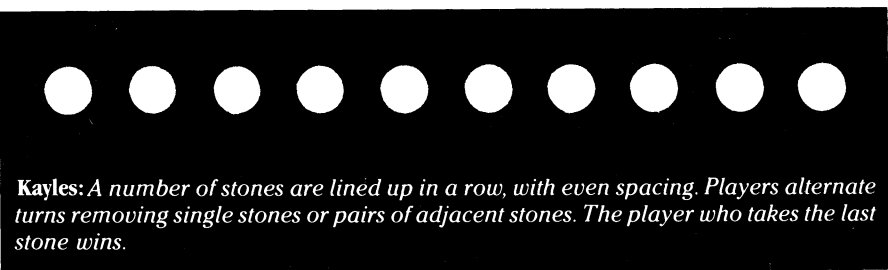


Traffic Jam: *The game board represents one-way roads between towns. A certain number of vehicles are placed on the map, and players take turns driving from their present locations to adjacent ones, one move per turn. Play continues until a traffic jam develops, preventing any movement. The last player who moves wins.*

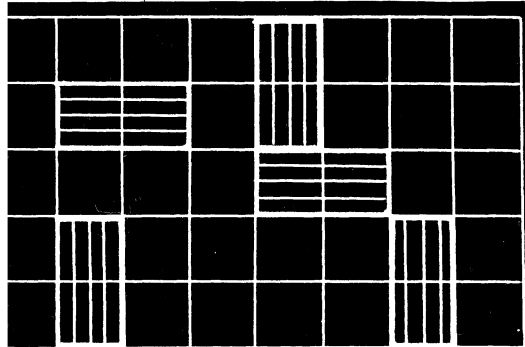


Two Dimensional Nim: *Tokens are placed on each spot in a rectangular or triangular array. Players alternate turns removing any number of tokens that all lie in a straight line. The player who makes the last move is the winner.*

tem in which, for instance, $6 + 3 = 5$, and $6 + 5 = 3$. (In binary notation 6 is 110 and 3 is 11; thus $6 + 3$ is, under Nim addition, 101, which is just $4 + 1$, or 5, while $6 + 5$ — that is, $110 + 101$ — is 11, which is $2 + 1 = 3$.) The basic rationale for this number system has been refined by John Horton Conway of Cambridge University to provide a remarkable new basis for under-



Kayles: *A number of stones are lined up in a row, with even spacing. Players alternate turns removing single stones or pairs of adjacent stones. The player who takes the last stone wins.*



Rectangular Dominoes: *Play takes place on a square or rectangular board (like a checkerboard) with domino pieces that exactly cover two squares. One player places them vertically, the other player places them horizontally. Dominoes may not overlap. The winner is the player who moves last.*

standing and axiomatizing the concept of "number." Conway's theory, first published in a 1976 fun- and pun-filled monograph *On Numbers and Games* (Academic Press), offers a unified structure (based on moves in games like Nim) for what he calls "All Numbers Great and Small." A major recent article in the July 1977 *AMERICAN MATHEMATICAL MONTHLY* contains further Conway analysis of the relation between games and mathematics.

Conway's numbers include, in one structure, not only the ordinary whole numbers and fractions, but the general classes of irrational numbers, of infinite ordinal numbers and of infinitesimals. The discoveries that led to creation of these latter classes — by Richard Dedekind in 1872, George Cantor in 1880 and Abraham Robinson in 1965 — represent singular

achievements in our conquest of the world of numbers. Now, in a vivid gesture of mathematical imagination, Conway has shown how basic game ideas lead to essential but elusive general classes of numbers.

The key to the connection between games and numbers lies in the recognition that some games are really *sums* of simpler games. Nim, for instance, can be thought of as a sum of three copies of the simple (indeed trivial) game of taking stones from a single pile. This latter game is no contest: The first person takes all the stones and wins! But if you set on the table three versions of that game and then stipulate that at each turn a person can play in just *one* of the games, you have what Conway calls a compound game. You can do this with chess or checkers too: In compound chess it is possible for one player to make consecutive moves on a single board while the other player is taking consecutive turns on other boards. It is easy to see how radically this compounding can change the strategy of the original game. Conway saw in the implicit sophistication of the compounding process a mechanism for creating complicated number classes out of simpler ones.

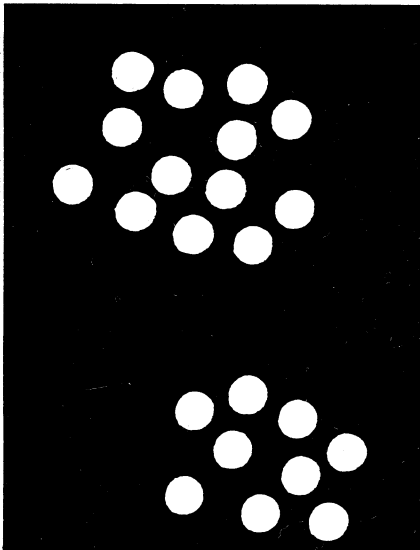
The game of Kayles is a good example of a sophisticated compound game. Here stones (or ninepins, formerly called kayles) are lined up in a row. On each turn a player (with an accurate ball) can take out any single stone or any adjacent pair of stones. As usual, the player who takes the last stone wins. This game begins as a single game, but as play proceeds it disintegrates into a compound of several smaller games.

Games like Kayles may be analyzed by a sequence of numbers called Sprague-Grundy (or simply Grundy) numbers that are analogs of the binary sums used to evaluate Nim positions. The Grundy numbers for Kayles begin as follows:

| | | | | | | | | | |
|----------------|---|----|----|----|----|----|----|---|---|
| No. of Stones | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| Grundy Numbers | 0 | 1 | 2 | 3 | 1 | 4 | 3 | 2 | 1 |
| No. of Stones | 9 | 10 | 11 | 12 | 13 | 14 | 15 | | |
| Grundy Numbers | 4 | 2 | 6 | 4 | 1 | 2 | 7 | | |

Although the pattern looks unpredictable, Richard Guy of the University of Calgary has shown that after 72 terms the Grundy numbers begin to repeat with a cycle of 12.

In 1907, W. A. Wythoff invented a variation of Nim that leads to still other interesting number patterns. Wythoff's game (sometimes called *Tsianshidsi*, after a similar ancient Chinese game) is played with stones in two piles of any size. Each player in turn can take any number of stones from a single pile, or else an equal number from both piles. (As usual, the player taking the last stone wins.) The safe patterns of piles in this game turn out to be (1,2), (3,5), (4,7), (6,10), (8,13), (9,15)... If, for instance, you leave your opponent facing piles of 6 and 10 stones, then no matter what happens you can always respond with a legal move that will leave one of the three lower safe configurations.



Tsianshidsi: Play begins with two piles of stones of any size. Each player in turn can take any number of stones from a single pile, or any equal number of stones from both piles. The winner is the player who takes the last stone.

The interesting thing about the Wythoff strategy is that the sequence of safe positions includes each positive whole number somewhere, but in an unpredictable pattern. We can see this more readily by making two lists of the numbers involved in the strategy for Wythoff's game, one for the first pile and one for the second:

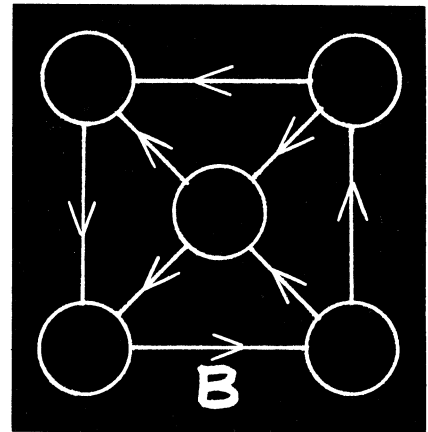
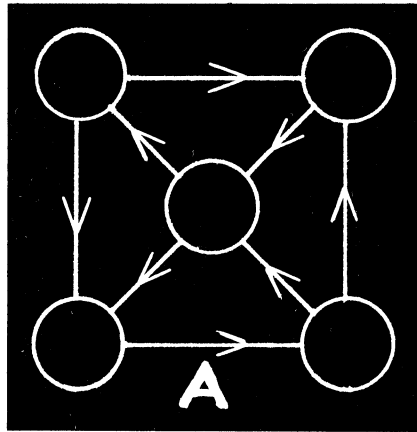
- 1, 3, 4, 6, 8, 9, ...
- 2, 5, 7, 10, 13, 15, ...

Mathematicians call sequences like these complementary, because each complements the other by providing the missing integers.

Investigation of the relation between complementary number sequences and strategies for games led mathematician Aviezri Fraenkel in Israel to create games of a new type that he called "annihilation games." In these games players move pieces (all of the same color) following one-way arrows on a game board, with the general rule that when a piece moves to an occupied location, both pieces involved are "annihilated" — removed from the board. Unlike games like checkers, annihilation games permit any player to move any piece. The complexity of even simple games of this sort is astounding — and is the subject of considerable current research in combinatorial mathematics and theoretical computer science.

Fraenkel's Innocent Marble Game is a good example, and a good game. It uses two basic game boards that are just slight variations of each other (see above).

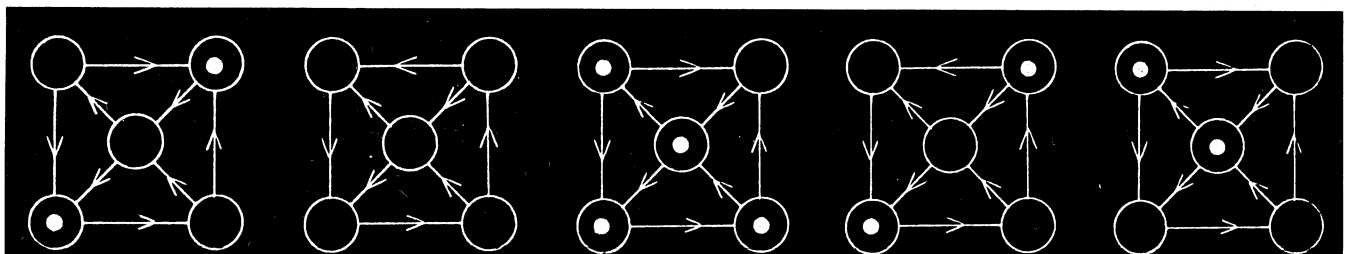
The normal game is a compound game using three copies of Board A and two copies of Board B. Each board contains an even number of marbles, no more than one in each pit. Players move any single marble on each turn, following the direction of the arrows. Marbles are removed in pairs by annihilation. The player who makes the last move wins.



Games like the Innocent Marble Game differ from Nim-like games in a fundamental detail: The marble game has loops that permit infinite cycling in which one marker chases another endlessly around in circles. The presence of these cycles led Fraenkel to devise a generalized Sprague-Grundy function that copes adequately with cycles. This new tool has permitted detailed analysis of a large class of games that, like the marble game, can have "dynamic ties" — unending infinite repetitions. Examples of such games are explained in an article by Fraenkel that appeared in the January 1978 *MATHEMATICS MAGAZINE*.

Research on games has yielded new insight into the important class of mathematical and decision problems called NP complete. These problems (SN: 5/8/76, p. 298) are very hard: The time required to solve them grows quickly beyond the reach of even the fastest modern computer. Fraenkel and other computer scientists have shown that some of the annihilation games are what is called NP hard: Determination of the best move in these games is at least as hard as any NP complete problem.

Compound games, defined by only a few very simple rules, can therefore simulate the most difficult problems in finite (or combinatorial) mathematics. They form an ideal proving ground for controlled investigation of more elaborate (and more "realistic") problems. Moreover, playing these games can be fun, with or without a thorough analysis of strategy. □



Innocent Marble Game: Zero, two or four marbles are placed, one per pit, in each of the five components of the gameboard. Players alternate turns, moving one marble one step at a time in the direction of the arrows. Whenever a marble is moved to a pit already occupied by another marble, both are removed from the board. The player who makes the last move is the winner.