★ A PLACE IN THE ★ SUN ★★★ FOR ★★★ FRACTALS

Those mathematical hybrids with fractional dimensions now are being used to analyze turbulence and the sun's power spectrum

BY DIETRICK E. THOMSEN

Fractals started out as a geometrical curiosity. Fractals are curves and surfaces that live in an unusual realm between the first and second or between the second and third dimensions. In the last few years, as they have been more and more widely publicized, fractals have gained more and more applications, from mapping (SN: 8/20/77, p. 122) to music (SN: 3/22/80, p. 187) and now to plasma physics and astrophysics. Some recent progress in the application of fractals to the theory of thermonuclear plasma turbulence and to the spectrum of the sun and the configuration of bodies in the solar system was reviewed at the recent meeting in New York of the Plasma Physics Division of the American Physical Society by Shane Johnston of Columbia University.

The most visible characteristic of fractal figures is the quality called self-symmetry or self-simulation. That is, a short portion of the figure looks like a fine-scale copy of the shape of a larger portion. This is often illustrated by the Koch curve (named for the German mathematician Helge von Koch). The Koch curve starts as a six-pointed star. The first degree of scale refinement consists of erecting 12 new points smaller than but similar in shape to the six existing points. One of these new points is erected on the middle third of each of the 12 sides that form the six original points. The next refinement is to erect new, smaller points on each of the many sides the figure now has. The refinement after that erects still more points, and so on, literally *ad infinitum.*

One of the curious properties of fractals is now evident: With each refinement the total length of the line in the figure increases, but the span of the space it occupies does not change much. The area enclosed by the Koch curve doesn't fluctuate greatly even if the refinement of scale or the degree of resolution — "increasing the resolution" is another way to say "refining the scale" — is driven to an infinite degree and the line becomes infinitely long. The east coast of the United States has fractal qualities: The length of the line drawn on a gross-scale map is less than that on a fine-scale map because the fine-scale map traces out bays and estuaries that the line on the gross-scale map crosses at their mouths. Yet the distance "as the crow flies" between Machiasport and Port Everglades stays the same.

For fractals, therefore, length depends on scale. But even if the resolution is taken to be infinite, a fractal does not connect up all the points in its part of space and become a two-dimensional surface as one would expect an infinitely long line in a bounded part of the plane to do. There are
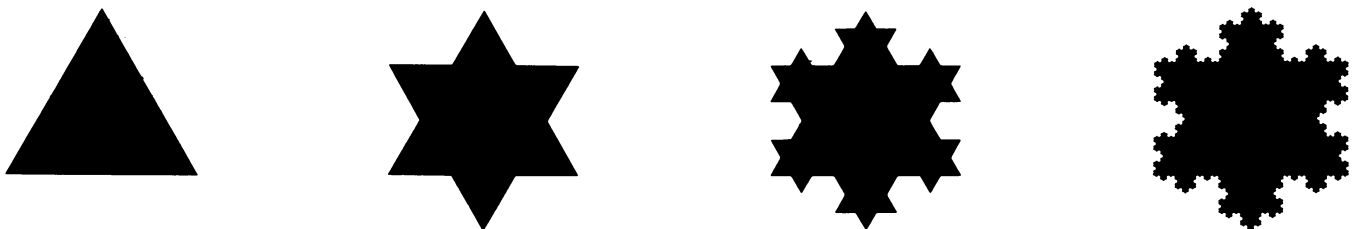
Successive scale refinements complicate the wiggles of the Koch curve, a representative fractal. Natural figures that are analyzed fractally, such as coastlines or musical scores, are equally wiggly but not as regular as the pure geometric shapes.

Mandelbrot

Fractals

still points in the space enclosed by the Koch curve, for instance, that are not part of the curve. A fractal is thus not really one-dimensional and not really two-dimensional. It hovers in a never-never land somewhere between. For fractals "length" had to be redefined, and a formula was developed for the so-called Hausdorf length that gives a measure of distance independent of the wiggles that keep increasing as the scale gets smaller. "Dimension" has to be regarded in such a way that fractions are permissible, and each fractal curve gets its own number. The dimension of the Koch curve, for example, comes to about 1.26.

Fractals were first applied in a number of situations where there are spatial or temporal characteristics reminiscent of the qualities of fractals: mapping, musical composition, the bonds and bends of macromolecules. The applications Johnston reviewed, turbulence in thermonuclear plasmas and the power spectrum of the sun, represent a move to the analytical side of mathematics rather than the directly geometrical.

Analysis deals with numbers and the elements that go with numbers to make up expressions and equations that can be manipulated according to the rules of arithmetic, algebra, calculus, etc. Analytical expressions are a less cumbersome and more versatile means of expressing and predicting relationships between different quantities than are geometric figures. Fractals can be expressed analytically in the form of series, Johnston says.

A mathematical physicist looks for an expression that has mathematical qualities analogous to the physical characteristics of the system for which a theory is sought. If such a mathematical expression exists, it may make a good description of the physics. If the mathematics can be shown to represent accurately the important numerical relationships in the physical system and to predict future changes in the important quantities, it may become the basis of a good theory of the physical system.

For turbulent motions in thermonuclear plasmas an equation with fractal characteristics may fit, what Johnston calls the "notorious Weierstrass equation," or rather an amended version called the Weierstrass-Mandelbrot equation. The Weierstrass-Mandelbrot equation connects the main quantities of interest in analyzing turbulent motion: frequency, amplitude and space. To give Johnston's definition, it is "the trigonometric sum of the geometric spacing of frequency with amplitude," all this expressed in numbers.

The solutions of the Weierstrass-Mandelbrot equation form a series of equations to which different fractal dimensions may be assigned. If these solutions are graphed, they come out spiky, full of sharp corners. They have a kind of accordion-pleat appearance. For these equations fractal dimension measures the degree of roughness, the number and sharpness of the abrupt changes in the quantity being calculated, which are represented by the spikes in the graph.

In these graphs, as Johnston points out, there's "never a straight line." That characteristic is highly reminiscent of turbulent motion: "It's intrinsically nonlinear," Johnston says. These solutions to the Weierstrass-Mandelbrot equation seem to apply to turbulence in other ways, Johnston says. They seem to be erratic, but they are actually governed by a very few control factors, and at some critical point they show a transition from smoothness to roughness that is reminiscent of the onset of turbulence in a physical system.

Frank Peseckis of Columbia University was thus inspired to suggest a new way of analyzing turbulence in plasmas. The old way is to separate the motion into two components, a slowly varying average part and a rapidly varying random part. Instead of separating into slow and fast, Peseckis suggests using smooth and rough as the criteria: A smoothly varying part and a fractally varying part. The steps that follow the separation are to calculate the development of the two parts and how they affect each other. A principal goal, says Johnston, is to calculate transport coefficients, that is, the rates of movement of particles and energy through the turbulent plasma. The transport coefficients would be determined by the fractal dimension of the turbulence at the particular location where someone wanted to know them. If it works, this could provide some handy rules for calculating how a particular degree of turbulence affects the characteristics of a plasma (temperature, density and confinement) that are crucial to fusion experiments. It's got a way to go till it proves itself.

Meanwhile, similar kinds of analyses are being applied to analogous systems, the power spectrum of the sun, for instance. The power spectrum relates the amount of power emitted by the sun at any given frequency to that frequency's place in the succession of frequencies. The quantities of interest here are similar to those of the turbulent plasma and the solar radiation is produced by a system that has many analogies to the turbulent plasmas studied on earth. Theoretical analysis of the solar spectrum is proceeding along similar lines.

Once fractals got into the solar system they were bound to extend the area of their influence. Johnston mentions that efforts are also underway to use fractals in developing a theory for the distribution of asteroids in the asteroid belt. This is a topic that traditional gravitational theory finds very difficult to handle. Fractals have been considered a far-out topic in mathematics. With astrophysics their applications are far out too. □