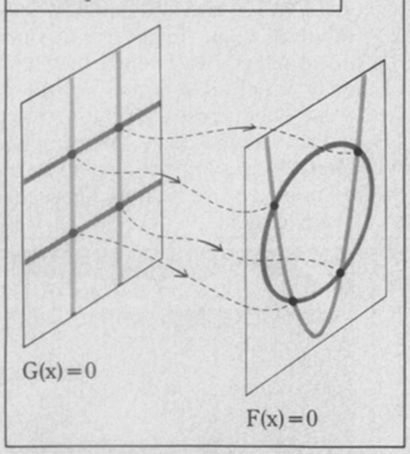
The Continuation Method

The need to solve systems of polynomial equations arises in pursuits ranging from geometric optics to chemical kinetics. A practical method of solution, developed at the General Motors Research Laboratories, provides designers of mechanical parts with a new capability.

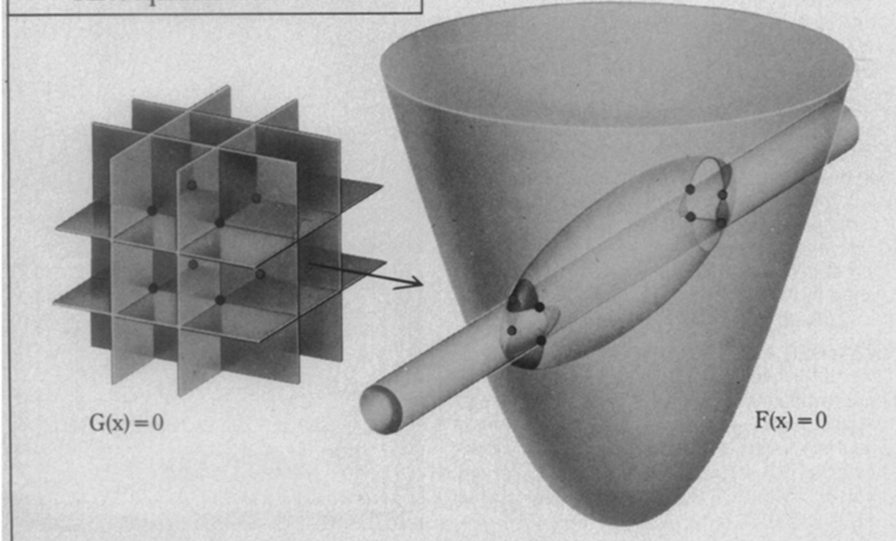
Two equations in two unknowns



The two pairs of parallel lines of G(x) = 0 evolve into the parabola and ellipse of F(x) = 0.

The three pairs of parallel planes of G(x) = 0 evolve into the paraboloid, ellipsoid and cylinder of F(x) = 0.

Three equations in three unknowns



CLASSICALLY difficult non-linear equations—those made up of polynomial expressions—can now be solved with reliability and speed. Recent advances in the mathematics of continuation methods at the General Motors Research Laboratories have practical implications for a wide range of scientific and engineering problems. The immediate application at General Motors is in mechanical design. The new method finds all eight solutions to three quadric equations in a few tenths of a second—fast enough for computer-aided design on a moment-to-moment basis. Algorithms based on this method are critical to the functioning of GMSOLID, an interactive design system which models the geometric characteristics of automotive parts.

Systems of non-linear equations have been solved for many years by "hit or miss" *local* methods. The method developed at General Motors by Dr. Alexander Morgan is distinguished by being *global* and *exhaustive*. Local methods depend on an initial estimate of the solution. They proceed by iterative modifications of this estimate to converge to a solution. However, success is not guaranteed, because there are generally no practical guidelines for making an initial choice that will ensure convergence. Reliability is further compromised when multiple solutions are sought.

Global methods, by contrast, do not require an initial estimate of the solution. The continuation method, as developed by Dr. Morgan, is not only global, but also exhaustive in that, assuming exact arithmetic, it guarantees convergence to all solutions. The convergence proof rests on principles from the area of mathematics called differential topology.

Here is the way continuation works. Suppose we want to solve a system F(x) = 0. We begin by generating a simpler system G(x) = 0 which we can both solve and continuously evolve into F(x) = 0. It is important that we select a G properly, so the process will converge. Dr. Morgan has devised a method for selecting G which gives rapid convergence and reliable computational behavior. He first applied a theorem established by Garcia and

Zangwill to select G. However, the resulting algorithm could not achieve the speed and computational reliability necessary for several applications. Next, he utilized some ideas from algebraic geometry—"homogenous coordinates" and "complex projective space"—to prove a new theorem for selecting G. The result of Dr. Morgan's efforts is a practical numerical method based on solid mathematical principles with innate reliability.

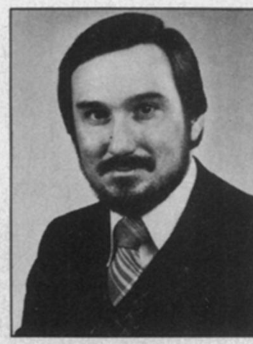
Reliability is the critical element for mathematical methods embedded in large computer programs, because errors may not become evident until after they have ruined a large data structure compiled at great expense and effort. Speed is also important to economical real-time implementation. This method has proved to be reliable and fast in solving problems involving equations up to the sixth degree in three or four variables. However, there are obvious practical limitations on the number of equations and their degree, due to the limited precision of computer arithmetic and computer resource availability.

THE FIGURES illustrate the transition from simple $G(x)=0$ to final $F(x)=0$. In both figures, the "simplicity" of $G(x)=0$ is reflected graphically in its linear structure—seen as lines and planes. The non-linearity of $F(x)=0$ is seen in the curvature of the final shapes in each figure.

In figure 1, the four dots on the left plane represent the set of simultaneous solutions to the system of equations $G(x)=0$. The four dots on the right plane represent the set of simultaneous solutions to the system of equations $F(x)=0$. The dashed lines represent simultaneous solutions to intermediate systems whose graphs would show the evolution from one configuration to the other. With the addition of a third dimension in figure 2, the number of dots representing simultaneous solutions doubles. Representation of the transitional points, as in figure 1, would require a fourth dimension.

"Continuation methods, although well known to mathematicians," says Dr. Morgan, "are not widely used in science and engineering. Acoustics, kinematics and non-linear circuit design are just a few fields that could benefit immediately. I expect to see much greater use of this mathematical tool in the future."

THE MAN BEHIND THE WORK



Dr. Alexander Morgan is a Senior Research Scientist in the Mathematics Department at the General Motors Research Laboratories.

Dr. Morgan received his graduate degrees from Yale University in the field of differential topology. His Ph.D. thesis concerned the geometry of differential manifolds. Prior to joining General Motors in 1978, he taught mathematics at the University of Miami in Florida and worked as an analyst at the Department of Energy's Savannah River Plant in South Carolina.

While serving in the U.S. Army, Dr. Morgan participated in the development and analysis of simulation models at the Strategy and Tactics Analysis Group in Bethesda, Maryland.

Dr. Morgan's current research interests include the qualitative theory of ordinary differential equations and the numerical solution of non-linear equations.