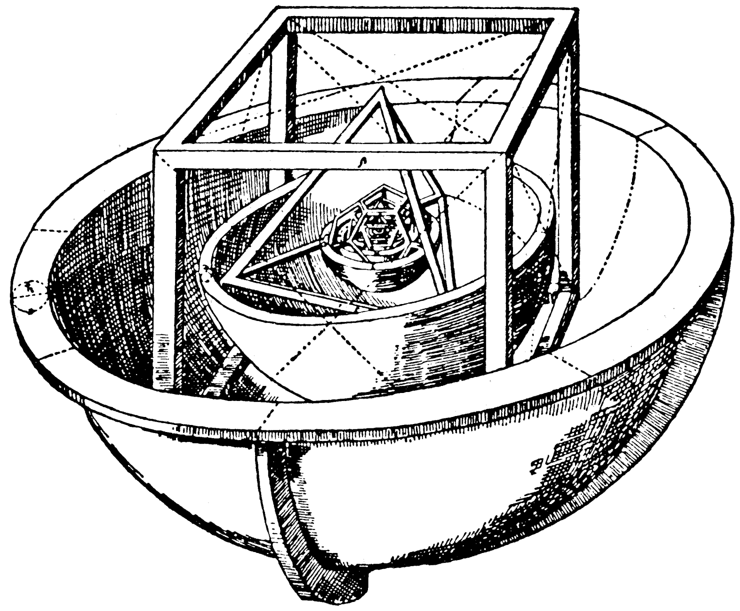
Twisting and Turning In Space

Classifying shapes and surfaces in new dimensions suggests the possibility of at last achieving a unified geometry



Physical Thought, S. Sambursky, Ed. Pica Pr. 1974

The Mathematical Experience/Birkhauser

Henri Poincare'

By LYNN ARTHUR STEEN

From the metaphysics of ancient Greece to the dawn of the scientific revolution, certain perfect forms — the circle, the sphere, the five Platonic solids—served as metaphors for the way the world works. The unending cyclic repetition of the circle explained the seasonal patterns of nature. Epicycles, formed from rolling circles on the hemisphere of the sky, explained the motion of the heavenly bodies. And as Kepler tried to show, the geometry of nested Platonic solids (see diagram above) explained spacing among the five inner planets of our solar system.

But nature's symmetry is more subtle than the Greek geometers ever suspected. Today we recognize these metaphors more as myths than as models. Yet an attempt to represent nature by geometric models persists. First, elliptical orbits replaced circles and epicycles in planetary astronomy. Then four-dimensional space-time replaced three-dimensional Euclidean space in models of the universe. Now, twisted surfaces of interlocked time, energy and space are being used to model black holes and the big bang.

Even the social sciences have taken up geometric models. Representation of public opinion requires multidimensional constructs. Models of human perception employ bizarre, non-Euclidean geometries. Even economic models routinely use infinite-dimensional models in an attempt to represent the unbounded variation of real economies.

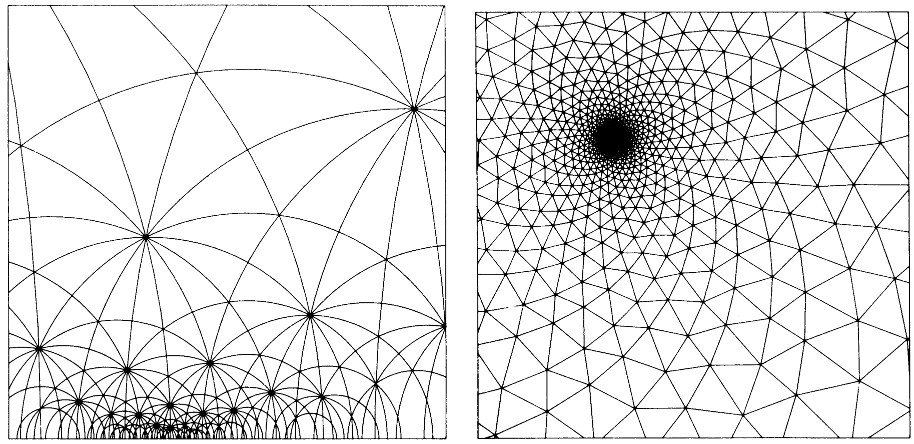
This proliferation of scientific uses for geometric models has been matched, however, and probably exceeded by mathematicians' exploration of geometric structures. While the Greek penchant for abstraction and classification continues,

What's Euclidean

On a perfectly flat surface, parallel lines never meet. A triangle drawn on such a plane has interior angles that add up to 180°. A straight line is the shortest distance between two points. These and other postulates and results are part of the geometry first invented by Euclid more than 2,000 years ago. Later mathematicians refined and extended his ideas to higher dimensions, and the postulates became part of what is now called **Euclidean geometry.**

However, surfaces exist on which the Euclidean rules do not hold. On the surface of a sphere, for example, the shortest distance between two points is an arc (called a great circle) following the sphere's curvature. A triangle bounded by "straight" lines has angles that add up to more than 180°. What appear to be parallel lines intersect in antipodal points. Because the geometry on a sphere is different from that on a flat surface, it is one example of a non-Euclidean geometry.

Manifolds are surfaces and shapes, sometimes very complex, that appear to be Euclidean when a small region is examined, but on a large scale fail to follow the rules for a Euclidean geometry. A sphere is an example of a manifold because a small region of the sphere appears to be flat, although on a larger scale the surface is clearly curved. *– I. Peterson*


Thurston/Bull. of the Am. Math. Soc.

Geometric constructs for modeling three-dimensional manifolds.

the domain of discourse has been enormously extended — into higher dimensions, into topological (rubber-sheet geometric) models, into curves, surfaces and diverse shapes called manifolds (see box, p. 42).

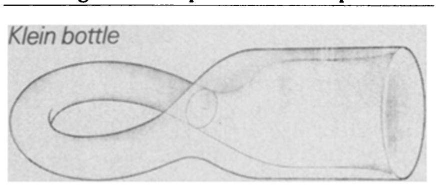
Roughly speaking, our discovery of new geometric objects has generally exceeded our ability to classify them. (It's as if Darwin, rather than being a passenger on the earth-bound *Beagle,* had been a passenger on Starship Enterprise, discovering on each new world exotic forms that defied easy classification.)

Now, at long last, mathematicians' efforts to classify geometric shapes may be approaching an end. One of the major unresolved sections of this multidimensional puzzle was just completed, and the outlines of the remaining pieces are beginning to take shape.

The origins of this scheme, however, stretch back to the turn of the century. Geometry's Darwin was the incomparable French mathematician Henri Poincaré (1854-1912), the one person who almost beat Einstein to the theory of relativity. In studies conducted around 1900, Poincaré explored many special surfaces and volumes as potential domains for the solutions to differential equations.

Solutions of some differential equations can be readily represented by paths taken in ordinary two- or three-dimensional Euclidean space (see box, p. 42). Other equations are best modeled by solutions on a sphere, a torus (doughnut-shape), or even more exotic objects — such as a twisted loop with only one side, known as a Möbius strip (see p. 36); or a three-dimensional tube, known as a Klein bottle, whose inside surface loops back on itself to merge with its outside. Poincaré set out to classify these many forms and surfaces, and in so doing helped create the modern field of topology.

In geometry, notions such as distance and angle are of paramount importance.


Klein bottle

But what counts in topology is whether one shape can be continuously deformed into another. All five Platonic solids, for example, are topologically equivalent not only to each other, but also to a sphere. However, because it has a hole in it, a torus is quite a different topological object.

Poincaré discovered an algebraic way to detect and study the pattern of holes in a topological surface. His strategy in this "algebraization" of topology was to examine the behavior of simple closed paths (those which, like a circle, begin and end at the same point) on the surface. All closed paths on a sphere, for instance, can be continuously shrunk to a single point, whereas only some of those drawn on a torus can be shrunk to a point. That's because on a torus there are paths of fundamentally different species: Those winding *through* the center hole are quite different from those winding *around* the center hole. And both are different from the cyclic paths that loop through the center as they go around the torus.

Poincaré showed that species of paths on surfaces form a group — a simple, algebraic concept that represents geometric symmetry. And the nature of a fundamental Poincaré group tells a lot about the nature of its corresponding surface. This representation of topological surfaces by Poincaré groups is now part of the theory of *homotopy* (literally "of the same shape") — so named because it represents the mathematical transformation of one curve into another, or the shrinking of a curve into a point (when that is possible). Once the exclusive tool of algebraic topologists, homotopies have recently come to be employed by graphic artists and computer scientists who wish to provide smooth videotape transitions between frames of artistic animation.

Poincaré's fundamental homotopy group is sufficiently discriminating to identify all two-dimensional surfaces: Two such surfaces are topologically equivalent if (and only if) they correspond to the same Poincaré group. In the third dimension, however, things turn out to be less simple. In fact, Poincaré's effort to extend homotopy classification to three-dimensional objects ended where it began — with the simplest case, the sphere.

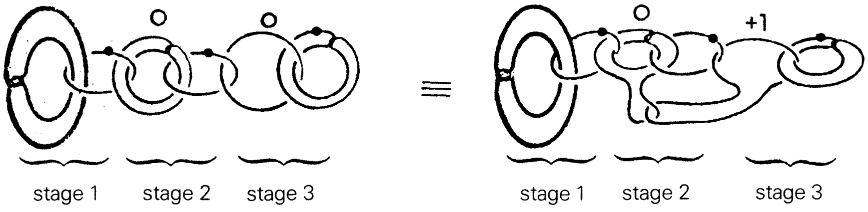
Poincaré conjectured that three-dimensional spheres behave just like two-dimensional ones — that every three-dimensional object that had the same homotopy groups as the sphere is topologically equivalent to a sphere. In other words, there should be no fake spheres: An object behaving like a sphere should be a sphere. Poincaré's conjecture and its generalization to higher dimensions ranks as one of the most challenging unsolved problems in mathematics. And it's worth noting that whole fields of mathematics have developed in the course of testing, on a case-by-case basis, the Poincaré conjecture.

Topologists quickly discovered as they explored geometric structures in higher dimensions that neither the intuition nor the vocabulary of simple geometry would suffice. So they introduced the "manifold" as a general term for certain topological objects that (in any dimension) include a great variety of surfaces and spaces. The great task of topology in this century has been to classify manifolds — to discover the origin of these exotic geometric species.

The first major advance came in 1954, when Poincaré's countryman, René Thom, discovered an important clue to the organization of manifolds: When two manifolds together serve as a boundary for a third, they share important related characteristics. Thom's theory, called "cobordism," provided insight into ways manifolds behave in dimensions greater than four.

In 1962, Stephen Smale of the University of California at Berkeley used an extension of cobordism techniques to prove the Poincaré conjecture for manifolds of the fifth and higher dimensions. Shortly after, John R. Stallings and E. Christopher Zeeman provided alternate proofs of the same results. (Ten years later Thom and Zeeman used ideas derived from this work on classification of manifolds to investigate and classify "elementary catastrophes" — a subject that spawned widespread interest and controversy when it was applied to research in the behavioral sciences [SN: 4/2/77, p. 218].)

Manifolds in higher dimensions (those above four) are in many respects paradoxical: Their classification proved easier



Freedman

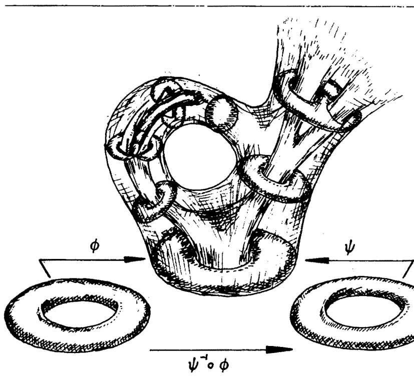
An example of handle body calculus.

than did those of dimensions three and four; however, even the simplest manifolds — spheres — would sometimes exhibit certain bizarre behaviors in high dimensions. For example, in 1956 John Milnor of Princeton University stunned the mathematical world by showing that the seven-dimensional sphere can be made into a "differentiable" manifold in 28 different ways. In other words, Poincaré's scheme of representing the solutions to differential equations on manifolds would, in dimension seven, encounter 28 different versions of the sphere.

Milnor constructed these exotic spheres by multidimensional plumbing — cutting holes here and there, connecting them by various tubes, and then deforming what resulted into a new kind of sphere. This procedure, called surgery, has proved as valuable as cobordism in the general task of classifying manifolds. And its modern form leads to what is called "handle-body calculus," a scheme for computing the effects of adding handles to manifolds. (If the handles are interlocked, or tied in knots, things can get rather complicated.)

Since the 1960s, efforts toward proving Poincaré's conjecture have been stalled — at the point where all but the last two cases had been verified. Now Michael Freedman of the University of California at San Diego has completed work on proving the next to last case — the fourth dimension. Strangely, the last unproved case is the third dimension — the same one that stumped Poincaré.

Freedman's work concerns four-dimensional manifolds and is based largely on handle calculus. "The dream remained since my graduate school days," writes Freedman, "that some key principle from the high dimensional theory would extend, at least to dimension four, and bring with it the beautiful adherence of topology to algebra familiar in dimensions greater than or equal to five."



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An example of surgery.

"There is such a principle," continues Freedman. And in the parlance of mathematics he goes on to describe it as "a homotopy-theoretic criterion for imbedding a topological 2-handle in a smooth four-dimensional manifold with boundary." Freedman blends homotopy, handles and boundaries for a comprehensive analysis of the topology of four-dimensional manifolds that yields the first complete classification of manifolds in this dimension. Although Freedman's paper has not yet been published, it has been checked carefully throughout the past year by topologists at several major universities. Now, only the third dimension remains without a satisfactory classification, and without a proof of the Poincaré conjecture.

But this case, too, appears to be nearing completion. William P. Thurston of the University of Colorado, writing in the May 1982 Bulletin of the American Mathematical Society, sets forth a program for classification of three-dimensional manifolds, including resolution of the original Poincaré conjecture. Thurston's program uses surgery to reduce complex manifolds to simple cases in much the same way factoring reduces large composite numbers to products of primes.

Thurston's paper offers a direction for research, not a complete theory. However, the mathematician marshals a lot of evidence in support of this program, not least new and beautiful computer realizations of many of the algebraic and geometric constructs needed to model three-dimensional manifolds.

He uses the computer, for example, to portray the geometric structures resulting from surgery. "The geometric structures turn out to be very beautiful when you learn to see them," writes Thurston. "Often the information which determines a geometric structure can be expressed in terms of some construction in plane Euclidean geometry."

Two years ago mathematicians celebrated the successful completion of a century-long effort to classify "finite simple groups" (SN: 9/27/80, p. 204). Now they are nearing the end of a similar research effort concerning the classification of manifolds. The end may come next year. Then again, it may not be until the end of the century that the case for the third dimension is finally verified. When it is completed, however, both algebra and geometry — the two traditional branches of mathematics — will be governed by grand theories of classification and evolution, theories that relate fundamental objects to each other in a way that imposes useful structure where formerly only chaos reigned. □

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DIET, NUTRITION, AND CANCER — National Research Council, Committee on Diet, Nutrition, and Cancer, Clifford Grobstein, Chairman. Summarizes the most relevant scientific information on diet and cancer. Recommends dietary guidelines designed to reduce the risks of developing cancer. (See SN: 6/26/82, p. 422.) Natl Acad Pr, 1982, 496 p., paper, $13.50.

THE ENERGY ANSWER 1982-2000 — Richard C. Dorf. Discusses the energy dilemma in the United States, the effects of technological innovation, the development of existing energy sources and prospects for the next two decades. Brick Hse Pub, 1982, 115 p., charts & graphs, paper, $8.95.

THE FOSSIL RECORD AND EVOLUTION: Readings from *Scientific American* — Introductions by Léo F. Laporte. Provides an overview of evolution and history of life as recorded by the sequence of fossils preserved in the earth's crust. W H Freeman, 1982, 225 p., color/b&w illus., paper, $11.95.

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RESEARCH & DEVELOPMENT: Federal Budget — FY 1983/Impact and Challenge, AAAS Report VII — Willis H. Shapley, Albert H. Teich and Jill P. Weinberg. An examination of the proposals in the 1983 budget from the perspective of their effect on research and development in the public and private sectors. In the final chapter the budget process is explained and the steps in the process are outlined. (See SN: 7/3/82, p. 6.) AAAS, 1982, 159 p., paper, $8.