Even simple mathematical expressions can behave in unexpected ways and display patterns of startling beauty

$1/e \times e^z$ $\quad$ $c\,e^z,\ c > 1/e$

By IVARS PETERSON

ESCAPE INTO CHAOS

Dot by dot, the black screen fills with color: an iridescent dragon clawing at its own tail, a swirling, rainbow-hued galaxy scattering vivid sparks, geometric fountains spilling colored streams into still, black basins. Each picture represents a frame in a mathematical experiment.

To mathematician Robert L. Devaney of Boston University, the colors and patterns have special meanings. He is one of many mathematicians now using computers to explore the behavior of mathematical expressions. "I see a whole new branch of mathematics developing called 'experimental mathematics,'" says Devaney. "Most other sciences — physics, chemistry, biology — have very definite, well-entrenched experimental sides as well as theoretical sides. Now the computer is becoming the mathematician's laboratory."

Computer pictures "open whole new worlds to the theoretical side," says Devaney. Pure mathematics asks: Is it true or isn't it? Experimental mathematics suggests possible truths that can then be explored more rigorously and formally.

Devaney's explorations involve the simplest "transcendental" mathematical expressions: the exponential, sine and cosine functions. The exponential function, represented by e to some power x, is familiar to anyone dealing with compounded growth, whether in populations or in accumulated interest in a savings account at a bank. The sine and cosine functions (usually written as sin and cos) are often associated with angles and come up in numerous trigonometric applications such as navigation or surveying.

Interest in these simple functions arises partly because of the strange behavior of nonlinear differential equations that are used to describe or model fluid flow, the formation of weather systems and other natural events. For the past decade or so, many investigators have discovered that under the right conditions, these equations themselves seem to generate "chaotic" numerical results. At the same time, researchers have also realized that even the simplest natural phenomena at times appear to show chaotic behavior (SN: 7/30/83, p. 76).

"Inevitably, you want to study specific equations that arise, but very often these are just too complicated to understand," says Devaney. "So you are led inexorably to simpler and simpler systems. If you can't understand the exponential, sine and cosine maps [or functions], then you don't have any chance of understanding something more complicated."

In addition, Devaney notes, "Since the simplest possible models give chaotic behavior, one must assume that for complex models there would be even more complicated behavior, so that any physical system should exhibit some degree of unpredictability despite the fact that it's deterministic."

In his studies of the "dynamical" behavior of simple mathematical expressions, Devaney chooses to deal with "complex" numbers rather than ordinary "real" numbers. When complex numbers were invented centuries ago, no one could think of any practical uses for them. Now, they regularly show up in methods for solving differential equations and in other applications of calculus. They also play an important role in describing physical phenomena like electromagnetism and the properties of electrical circuits. As a result, it becomes important to know how the exponential, sine and cosine functions behave for complex numbers.

A complex number, z, is made up of a "real" part and an "imaginary" part. It may be written as $x + iy$, where the symbol "i" represents the square root of -1. These numbers can be plotted on a graph to produce what is called the complex plane. For example, the complex number $2+3i$ would be plotted at a point that is 2 units to the right of the vertical (imaginary) or y axis and 3 units up from the horizontal (real) or x axis. Thus, every complex number is located according to its coordinates somewhere in the complex plane.

The process of iteration, performing the same operation over and over again on successive answers, is the key to Devaney's colorful, computer-generated graphic designs. He selects a particular complex number z and calculates, for example, sin z. Then he calculates the sine of this answer and repeats the process for each new answer.

Depending on the value of z chosen, the same answer may come up every time (a fixed point), or the answers stay close to the original value or even return to the original value after a certain number of iterations (a periodic point). On the other hand, the answers may get steadily larger. In the latter case, Devaney assigns a specific color to the original point in the complex plane.

"We color a point in the plane if an iterate of that point ever has an imaginary part larger than 50 or smaller than −50," says Devaney. "So the colors tell me how 'quickly' a point goes to 'infinity.'" In his computer pictures, red represents points that explode beyond the limit in only one or two steps. The colors orange, yellow, green, blue and violet represent successively slower rates. Black areas encompass points that, upon iteration, map into values that do not escape.

The black areas, called basins of attraction, are stable regions. Devaney explains, "All points that are colored black, under iteration, tend toward fixed points or periodic points called attractors." The colored areas represent unstable, chaotic regions. For these values of z, the chosen function seems to behave randomly. "I'm interested in understanding the differences between stable regions (the black regions) and the colored regions," says Devaney.

The colored regions for a given complex function also give the "barest outline" of something called the Julia set (named after French mathematician Gaston Julia). This mathematical set contains all "repelling, periodic points" that seem to drive neighboring points farther and farther away. The collection of these special points corresponds to a "strange repeller." The complex plane thus divides into two intricately shaped regions: basins of attraction centered on "attractors" and Julia sets corresponding to "strange repellers."

The Julia sets that Devaney finds are also fractals (SN: 1/21/84, p. 42). Examine any of the patterns closely and one finds that their features tend to replicate themselves on smaller and smaller scales. A fist bursts into fingers that each burst into smaller fingers and so on.

Small changes in a function can radically change the form of the graphs. If the exponential function is multiplied by a constant factor, $1/e$, and then iterated, the resulting picture shows a small, sedate fountain within a large black basin. Make the constant slightly larger, and the picture changes dramatically. "The Julia set explodes from a relatively small piece of the plane into two spiralling galaxies," says Devaney. Similarly dramatic changes occur when sin z is multiplied by various values of a constant ranging from $1+.05i$ to $1+.8i$. As the imaginary part of the constant grows, the basin of attraction disappears.

"There are many complex analytic functions out there, all of which seem to have their own characteristic behavior," says Devaney. It would be useful to study a whole class of these different functions to get some idea of this behavior and then to extend the studies to higher dimensions, he says.

Ironically, the mathematics is proving to be so interesting that many of the mathematicians now working in the field are being led away from the physical applications that originally motivated the studies and away from trying to understand the roots of chaotic behavior in nature. "The process that made us study simple functions in the first place probably won't be reversed," says Devaney. "We're discovering so many new and interesting phenomena." These discoveries may eventually lead to entirely different, as yet unknown applications from those originally envisioned.

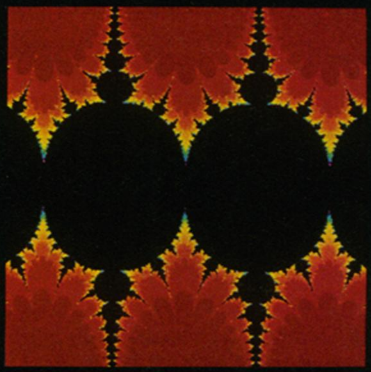
"It's really the computer that generates the mathematical problem," says Devaney. "You see something on paper, you try to explain it mathematically, but you can't. So you do more computer graphics, and it goes on like that."

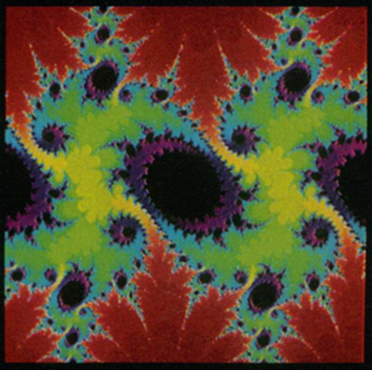
But is experimental mathematics a legitimate part of mathematics? Heinz-Otto Peitgen and his colleagues at the University of Bremen in West Germany, in describing their own computer graphics approach to exploring iterated functions and their Julia sets, write in THE MATHEMATICAL INTELLIGENCER (Vol. 6, No. 2, 1984), "Experimental mathematics will likely never be accepted as 'real' mathematics by most mathematicians. But for many enthusiasts it has become more than an engaging hobby — it is rather a passion. While such experiments will continue to enhance our mathematical intuition in the future, they might also develop into a sophisticated art form."

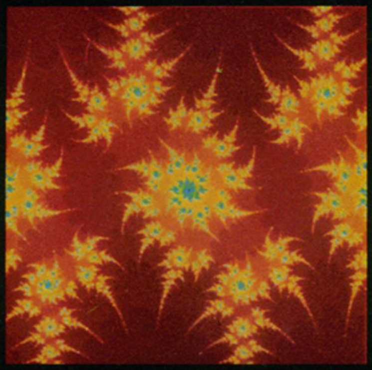
Computer experiments are bringing excitement and a new visual beauty to mathematics. Philip J. Davis and Reuben Hersch in their book *The Mathematical Experience* (Birkhäuser Boston, 1981) highlight this appeal. They write: "Blindness to the aesthetic element in mathematics is widespread and can account for a feeling that mathematics is dry as dust, as exciting as a telephone book.... Contrariwise, appreciation of the element makes the subject live in a wonderful manner and burn as no other creation of the human mind seems to do." □

Computer graphics by C. Small, C. Mayberry, S. Smith/Boston Univ.

c sin *z*







The top picture shows the iterated function sin z (c = 1). The left, right and bottom pictures show the pattern changing as c grows from 1 +.05i to 1 +.8i. The colors indicate how long it takes for iterates to "go far away": red = very quickly, then yellow, green, blue and purple = very slowly.