PLANNING BY COMPUTER: THE INSIDE ANSWER

By IVARS PETERSON

Lately, mathematician Narendra Karmarkar has been so busy answering the telephone that he is finding it hard to get back to his own research. The focus of all this fuss is Karmarkar's new method for solving problems in a field of applied mathematics called linear programming.

This planning technique, which helps establish the best strategy for keeping costs down, profits up or projects on schedule, is one of the largest users of computer time in business and industry. So, when word begins to spread that someone has invented a method that may speed up the process for finding answers to linear programming problems, the world comes calling.

Although theoretically sound and very exciting from a mathematical point of view, Karmarkar's scheme, nevertheless, has yet to prove itself in the business world to justify the stir that this news has created. So far, it has been tested only on a few linear programming problems, but the early results are encouraging.

"It was a very pretty theoretical result," says Michael Garey, Karmarkar's supervisor at AT&T Bell Laboratories in Murray Hill, N.J., "and I had some hopes that maybe for some fairly rare special types of linear programming problems it might be the way to go. But I never in my wildest dreams expected that it would win on such a broad front of problems, as it seems to be doing here."

"It's extremely clever," adds computer scientist Eugene L. Lawler of the University of California at Berkeley, one of many researchers beginning to study Karmarkar's method in detail. "It's really very different from anything that's been done in this kind of problem before," he says. "This is particularly suprising because linear programming has been so well worked over for years by so many people."

"It's a very, very exciting theoretical result," says George B. Dantzig of Stanford University, inventor of the widely used "simplex" method for solving linear programming problems. But he adds the caution, "It is not a practical result by itself." What isn't known yet is how quickly Karmarkar's method works for "real" problems like those encountered in business.

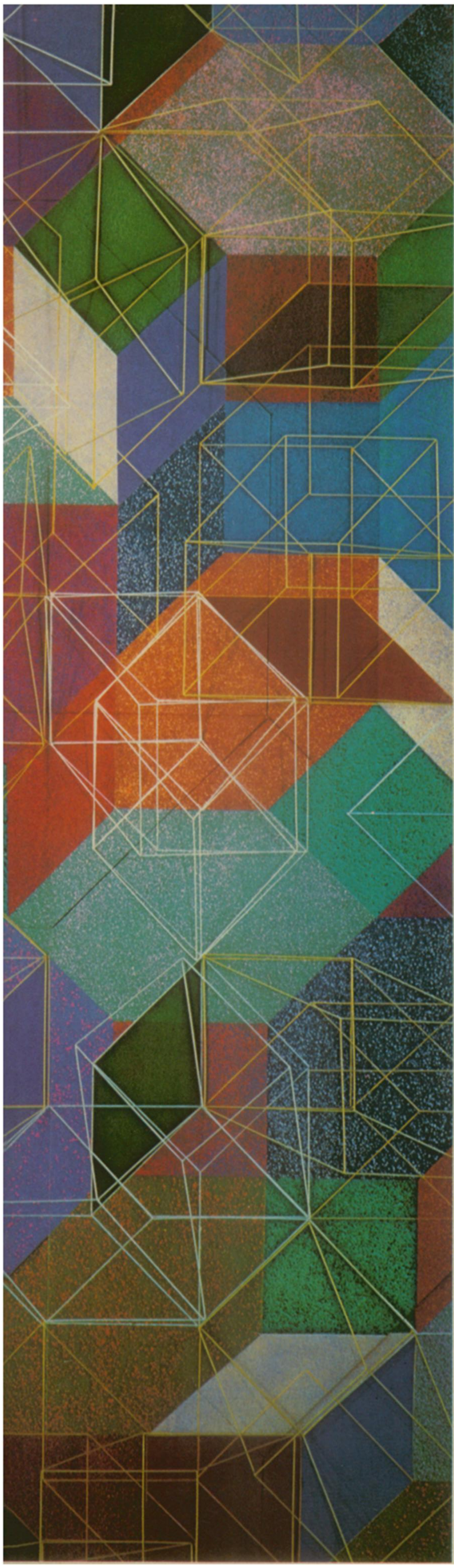
So far, Karmarkar is the only one who has tried his method on such problems. In one test, one of the fastest simplex programs available — a standard method in use for decades — took 50 times longer than Karmarkar's program to solve a problem containing about 5,000 variables. Although the results are encouraging, says Dantzig, Karmarkar may have been lucky in his initial choice of problems to solve.

"The issue is really how long it is going to take on typical practical problems," says Garey. "We won't really know that until we get some experience with it. I certainly hope that other people pursue the idea and try the algorithm too."

Linear programming (here, the word "programming" means "planning") is widely used for problems like the allocation of resources. One paper company, for example, increased its profits by $15 million in a single year by using linear programming to determine the optimum assortment of products that it should manufacture. The technique has also been applied to food distribution, oil refinery operations, company timetables and many other problems in engineering, economics and agriculture.

A simple, hypothetical example illustrates the principles involved. Suppose a small brewery sells ale at a profit of $10 per barrel and beer at a profit of $25 per barrel. However, the operation is limited by the availability of corn, hops and barley malt, and the recipes for ale and beer call for different proportions of the ingredients; for example, beer requires more corn per barrel than ale. At first glance, it appears that the brewer would make more money by brewing just beer. However, he may use up corn so quickly that a large quantity of hops and malt is left over and wasted. A better answer may be to brew both beer and ale so that as much as possible of the ingredients is used up, while still ensuring the maximum profit.

Linear programming was invented to solve this kind of conundrum. All of the constraints, such as the total supply of corn available, are expressed as linear equations ("linear" because doubling the amount of beer produced will double the



Robbin/Courtesy: General Electric Corporation Art Collection

In his work Four-field, *artist Tony Robbin uses polygons of color and metal rods that protrude from the canvas to create a vivid representation of a geometric fugue in four dimensions. Visualizing the multidimensional world of linear programming is many times more difficult, so mathematics takes over where pictures fail.*

amount of ingredients needed and so on). The aim is to find values, subject to the constraints, for the amounts of beer and ale that will generate the highest profit. Typical industry problems may contain thousands of variables and equations.

With two variables, all of the equations can be represented by straight lines on a two-dimensional graph. These lines intersect to form a closed geometric figure or a polygon. In general, the answers to a linear programming problem are the coordinates of one particular corner or vertex in that polygon. For problems with more variables, the figures are complicated multidimensional polyhedra (called polytopes) in high-dimensional spaces. In this case, the solution — the optimum values for all of the variables involved — lies at one vertex of the polytope. The trick is to locate that vertex. Fortunately, mathematicians have developed methods that usually avoid the necessity of visiting every vertex to find the best answer.

A new method for solving linear programming problems not only is a major theoretical achievement but also may speed up the search for solutions

The simplex method, introduced by Dantzig in 1947, finds the answer by hopping mathematically along an edge from one vertex to an adjacent vertex on the surface of the polytope, always taking the branch that leads "uphill" toward the "peak" vertex representing the optimum values. Theoretically, because the number of vertices can be immense, a computer using this method could take practically forever to reach the correct vertex. Nevertheless, in practice, the method is surprisingly efficient, and years of work have refined it considerably.

"One is really talking about an art," says Dantzig. "It takes about 20 lines to write down a program to say what the simplex method is. Actual programs have more than 50,000 lines. A lot of people have put all kinds of heuristics into the programs over the years."

The simplex method, however, sometimes gets stuck, and the size of problems that it can handle is limited. In 1979, Russian mathematician L.G. Khachian introduced a method that seemed to overcome some of these difficulties (SN:10/6/79, p.234). His "ellipsoid" method requires the construction of high-dimensional ellipsoids (in two dimensions, they would be ellipses) that "slide" sideways under the influence of the polytope's constraining surfaces so that the centers gradually converge on the solution. This procedure guarantees that a computer will complete its job within what scientists call "polynomial time," whereas the simplex method, at its worst, runs in "exponential time."

The following example illustrates the difference. If the time required for a computer to solve a problem shows polynomial growth according to, say, the cube of the problem's size, then increasing the number of variables from 20 to 50 increases computing time from, perhaps, 1 second to 12 seconds. On the other hand, if the algorithm grows exponentially, say, as 3^n, then a one-second solution becomes a thousand-century nightmare.

However, the Russian algorithm proved disappointing because, in practice, it turned out to be far slower than the simplex method for most everyday problems and had an advantage only when very large numbers of variables were involved. Except in certain pathological cases, the simplex method completes the job in about as many steps as there are variables. The worst case, in which every vertex must be visited, rarely comes up. Such shortcuts have not been found for the ellipsoid method.

Like the ellipsoid method, Karmarkar's scheme also runs in polynomial time, but the exponent governing how long his method takes is smaller than the exponent governing the ellipsoid method. As a result, the running time does not increase as rapidly for Karmarkar's method as it does for the ellipsoid method when the number of variables increases.

These bounds or "performance guarantees," which strictly limit how long an algorithm takes to do its job in the worst possible case, are important in theoretical computer science, says Hungarian mathematician Laszlo Babai, currently visiting the University of Chicago. Babai, who is studying the theoretical basis for Karmarkar's method, says establishing a new limit, by itself, makes Karmarkar's work a significant mathematical achievement.

Karmarkar does it by plunging into the interior of the polytope. This idea of working from the inside, instead of hopping from vertex to vertex, is not new, says Dantzig. "It's the first one that everyone thinks of." However, while almost all other attempts to invent a method using this approach have failed, Karmarkar succeeds by using projective geometry.

In his method, Karmarkar applies a mathematical operation that transforms the original space in which the polytope lies into the multidimensional analog of an equilateral triangle or a tetrahedron. The transformation also twists the polytope into a new shape. Once he finds any point that satisfies all of the constraints (although that point may not yet be the best one), he makes that point the center of his transformed polytope. From there, he can easily define a good direction in which to go to find an even better feasible point, around which the polytope is again reshaped. In relatively few steps, Karmarkar's method converges on the right vertex.

"It really contains ingenious ideas," says Babai, especially the use of projective transformations. "But this is a very new method, so there are many details that have to be worked out."

Now, Karmarkar's method, only a few months old, is at the beginning of a long period of testing and refinement. "Although it looks a bit mysterious at the present time, I'm quite sure that within the next few months, there will be a lot of simplification," says Lawler. "It will look a lot cleaner and easier for people to understand and work with."

"There are lots of improvements that one can make," says Karmarkar. He already knows of at least two dozen changes that will enhance his computer program. "The simplex method was developed and improved over almost 40 years," he says. "My method is still new." It was first published in April as part of an Association of Computing Machinery symposium on the theory of computing. An updated version appears in this month's issue of the mathematics journal COMBINATORICA.

"We have made an effort to put out the mathematics behind the method," says Garey. However, because Karmarkar's computer program implementing his method is still experimental, few people outside of Bell Labs are getting access to it. "We don't know what we're going to do with it in the long run," Garey says. "None of that has been decided." Some kind of licensing arrangement is one possibility.

"There are many problems that cannot be solved by the simplex method effectively," says Karmarkar. "Now, these problems fall within the range of what we can do." Says Babai, "Such a vast increase in the number of variables [that can be handled in a linear programming problem] would reshape many theories in economics and perhaps other sciences." Problems that people once avoided because there was no method for solving them now may become solvable. □