Three Bites in a Doughnut

Computer-generated pictures contribute to the discovery of a new minimal surface

By IVARS PETERSON

The iridescent shimmer and the smooth perfection of a soap film stretched taut across a wire frame has long evoked a strong fascination. This geometric surface, so graceful and economical, represents a soap film's physical solution to the mathematical problem of finding the smallest area that spans a given curve.

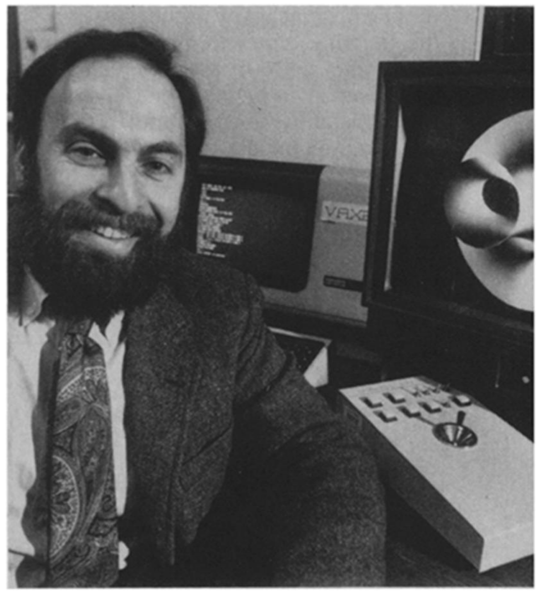
Over the years, mathematicians have extended this concept of a "minimal surface" to include forms that no soap film can take on. Says mathematician David A. Hoffman of the University of Massachusetts in Amherst, "The collection of all possible minimal surfaces is extremely rich, complicated and not yet completely understood." Now it takes a computer to draw some of the figures that come up in the study of these special surfaces.

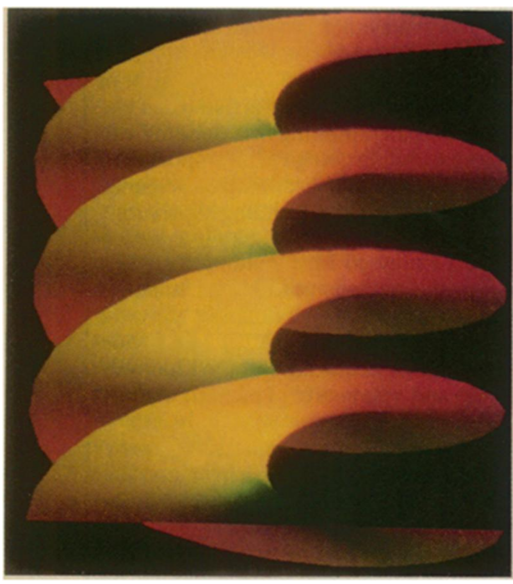
Recently, Hoffman contributed to the discovery of a new minimal surface, one that meets such tight mathematical conditions that it is the first of its kind to be found in almost 200 years. Computer graphics helped him visualize the surface. Then, he was able to prove mathematically that this surface has the required properties.

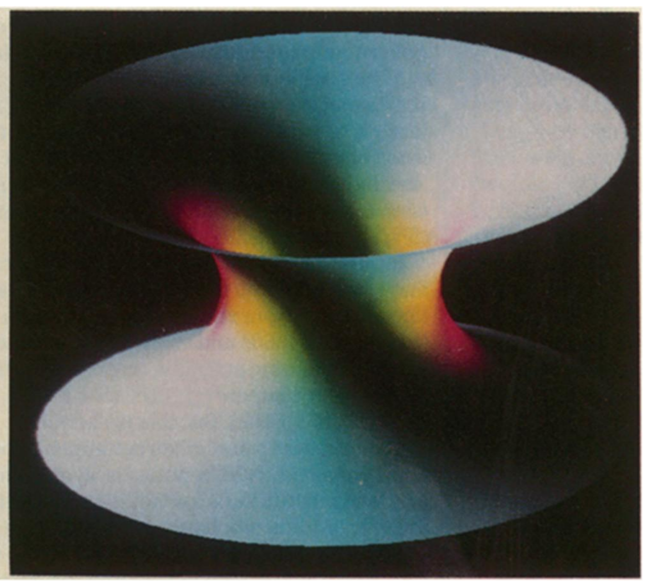
"We were surprised that computer graphics could actually be used as an exploratory tool to help us solve the problem," says Hoffman. "The surface couldn't be understood until we could see it. Once we saw it on the screen, we could go back to the proof."

The type of surface that Hoffman and his colleague William H. Meeks III of Rice University in Houston investigated is a "complete, embedded, minimal surface of finite topology." The term "complete" means that the surface, roughly speaking, has no boundaries. A smooth plane that extends in all directions without end is one example of a complete surface. It also happens to be a minimal surface because putting any kind of fold into the plane increases its surface area. Thus, an infinite surface can also be a minimal surface if each piece of the entire surface individually has the smallest possible area within the piece's boundaries.

Another example of a complete minimal surface, called the catenoid, looks like an infinitely extended hourglass. The soap film that connects two parallel circles of wire as they are pulled apart looks like the central piece of a catenoid. Both the catenoid and the plane are also "embedded" surfaces because they do not fold back and intersect themselves.







Computer-generated pictures helped David A. Hoffman (above) convince himself that he really had found a new minimal surface. This led to a mathematical proof that this figure had the right properties to qualify as a complete, embedded minimal surface. It now joins the plane, the helicoid (left) and the catenoid (right), which were the only previously known surfaces of this type.

Surfaces can also be classified according to their fundamental form or topology. Two surfaces have the same topology if they can be twisted, stretched or deformed in some way to convert one form to the other without tearing, cutting or gluing a surface. By this definition, a simple bowl and a ball have the same topology, while a doughnut (or torus) and a one-handled teacup represent the next, higher category. A pretzel, with two holes or "handles," goes another step farther.

In addition, each topological surface can have a number of "pinpricks" or "ends." A sphere with one such puncture point can be stretched out to form an infinite plane. This is somewhat like pulling on the rim of a narrow-mouthed clay pot to widen the pot's mouth and overdoing it in the process. A sphere with two puncture points can be deformed into the infinitely extended hourglass form of a catenoid.

Until the work of Hoffman and Meeks, the plane, catenoid and helicoid (imagine a soap film stretching along the curves of an infinitely long helix or spiral) were the only known examples of complete, embedded, minimal surfaces of finite topology. A few mathematicians had even quite recently speculated that these were the only possible examples.

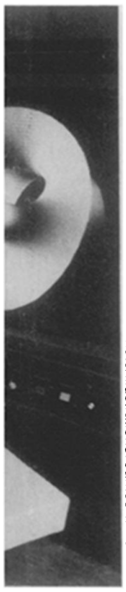
Nevertheless, says Meeks, "There were reasons for suspecting that these surfaces exist. If they were to exist, they would have certain symmetries, and from the symmetries one can derive the equations."

section, then he could go ahead with trying to prove that the surface really was embedded.

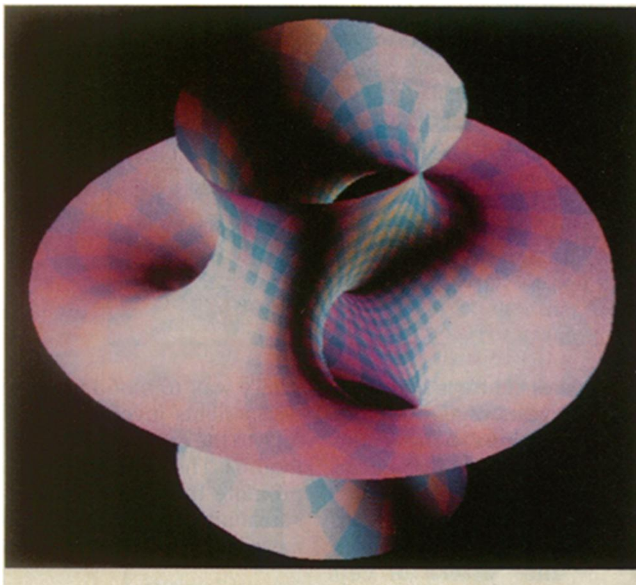
The first pictures did indicate that the surface was free of self-intersections. Seeing the surface from different points of view also showed that it had a high degree of symmetry, but it took "extended staring" to piece together the true form of this new minimal surface, says Hoffman. "How

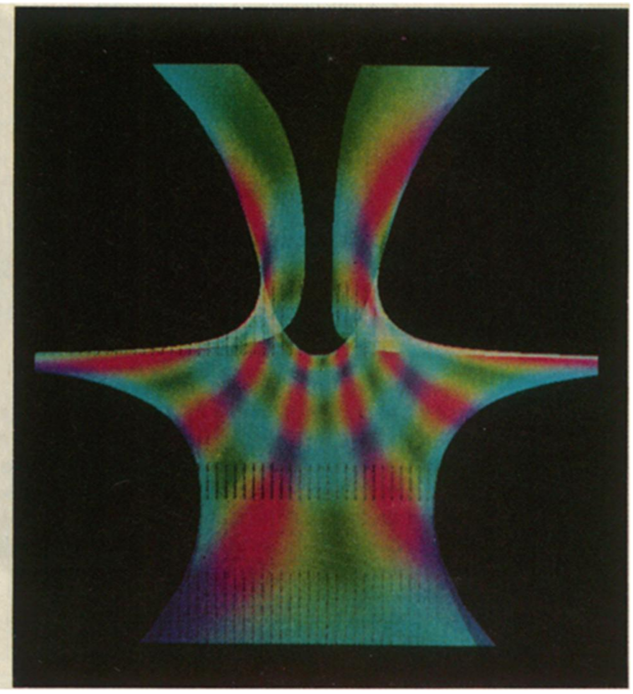
"ends." This work has opened up a new realm of geometric forms to explore.

"The computer was used in a situation where physical experimentation was impossible," says Hoffman. In the past, mathematicians and physicists sometimes worked with soap films to investigate minimal surfaces and the special differential equations for which these surfaces are solutions. Even today, some architects use



Photo/Illustrations: James T. Hoffman





The first, rough view (right) of what the equations looked like provided important clues about the nature of this new minimal surface, but it took views from many different angles before Hoffman could see its "true" form (above).

Hoffman started with the equations for a surface first written down by a Brazilian graduate student, Celso J. Costa, in his doctoral thesis. Costa was able to prove that this particular surface is minimal and complete. He also knew that topologically it is equivalent to a sugarcoated doughnut from which three bites have been taken. The bites indicate that the surface, when deformed, can extend to infinity in three places.

Meeks and Hoffman hoped that this surface would also meet the requirement that it be embedded. Mathematical clues indicated that the surface contained two catenoids and a plane that all somehow sprouted from the center of the figure. "We couldn't tell from the equations what was happening," Hoffman said at a recent meeting in Anaheim, Calif., of the American Mathematical Society. "It was hard to see what was happening in the middle."

With the help of the graphics programming skill of graduate student James T. Hoffman (no relation), David Hoffman computed numerical values for the surface's coordinates and drew pictures of its core. He knew that if he saw evidence that the surface intersected itself, then the surface would not be embedded and this particular mathematical quest would be over. If there was no visible evidence of an inter-

it fitted together was not obvious."

Hoffman puzzled over the computer pictures for days. The images had to fit together somehow. "All of a sudden, it came together, and all of the pieces fit," he says. "But after that, being able to convince others what it was like took still longer."

This new minimal surface, as revealed by computer graphics, has the elegance of a gracefully spinning dancer flinging out her skirt in a horizontal plane. Gentle folds radiate from the skirt's waist. Two holes pierce the lower surface of the skirt and join to form one catenoid that sweeps upward. Another pair of holes, set at right angles to the first pair, lead from the top of the skirt downward into the second catenoid.

The surface is made up of eight identical pieces that fit together to make up the whole figure, says Hoffman. It also contains a pair of straight lines. "These tips made it possible to analyze the equations...," he says. "This in turn led to a mathematical proof that this surface was in fact a new embedded, complete, minimal surface of a finite type, the first new one to be found in nearly 200 years." Later, Meeks and Hoffman showed that infinitely many examples of this type of minimal surface exist, one for each topological class with one or more "handles" and three

soap films and models to design sweeping, tentlike roofs for buildings. In many cases, however, soap films are unsuitable because the surfaces involved extend to infinity and the equations representing them are very complicated.

"To stretch the point a bit, computer graphics are to research on minimal surfaces 'in the large' what soap films are to research on minimal surfaces spanning a wire contour," Hoffman says.

More important, the computer was used as a guide in the construction of a formal proof. For a mathematician, seeing is *not* believing. Hoffman was able to explore ideas for proving the desired results by checking them first visually in a computed image. "Computer graphics functioned as an experimental tool in nailing down the proof," says Hoffman.

"The examples themselves are very beautiful," says Meeks. "They're very interesting because there wasn't much known about them." With these examples in hand and the possibility of inventing new minimal surfaces with more than three "ends," Meeks and Hoffman are now developing a general theory to describe the occurrence of these special minimal surfaces.

Hoffman adds, "This has also been the most fun of anything that I've ever done in my career." □