Untangling a Knotty Problem

Mathematicians find a new, simple way to distinguish different types of knots

By IVARS PETERSON

For mathematician Jim E. Hoste of Rutgers University in New Brunswick, N.J., it was an exciting event — a month and a half of solid work that unexpectedly led to a simple, new, theoretical way of looking at knots. "It was just amazing," he says. "Then I found out I wasn't alone."

Within a few weeks, five groups of mathematicians, including Hoste, reported identical results. "There are many instances of people making the same discovery at the same time," says Joan S. Birman of Columbia University in New York City. But the combination of so many different people with such a dramatic discovery makes this situation special.

Even more striking, perhaps, is that the proofs submitted by the various groups represent three genuinely different mathematical approaches, says David Yetter of Clark University in Worcester, Mass. Yetter and Peter Freyd of the University of Pennsylvania in Philadelphia came up with one of the proofs.

It illustrates the way in which different mathematical specialties are closely related, says Birman. "There really is just one mathematics," she says.

The discovery falls within the study of knots and links, part of a field of mathematics known as topology. In general, it's hard to tell if one particular knot tied in a string is mathematically equivalent to a seemingly different one tied in another string. One possible way to solve the problem is to try twisting one knot to transform it into the other.

It's somewhat like trying to decide whether scissors are needed to cut through a bundle of knitting yarn that seems to be hopelessly tangled, or whether the ends simply have to be pushed through carefully. "After a bit of twisting, you may be convinced that you can't do it," says Birman, "but maybe you weren't patient enough."

To a topologist, a knot in a formal sense is any simple closed curve embedded in a three-dimensional space. These knots, which have no free ends, can also be linked, like pieces of a chain, in complicated ways. Deciding whether two knots are equivalent, says Birman, turns out to be a deep question in topology.

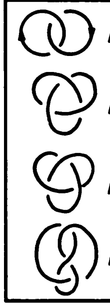
In 1928, J. W. Alexander of the Princeton Institute for Advanced Study discovered a systematic procedure for finding a characteristic algebraic expression to represent a particular knot. This "Alexander polynomial" is an example of a topological "invariant." If two knots have different Alexander polynomials, then the knots are definitely not equivalent.

But the Alexander polynomial doesn't supply the complete answer. Knots that have the same polynomial aren't necessarily the same. It doesn't distinguish, for example, between left-handed and right-handed knots.

Last year, Vaughan F. R. Jones of the University of California at Berkeley found a new polynomial that does a better job than the Alexander polynomial. "One of the main reasons why people were so interested in my polynomial," says Jones, "was that it so easily and in so many cases detected the difference between a knot and its mirror image."

The discovery took the mathematics community by surprise because Jones works in an area that is very far from knot theory. "He made a connection between operator algebras and knot theory," says Birman, whose work on "braids" provided an important link. "That was astonishing."

News of the Jones polynomial set off a wave of mathematical activity. "Everybody who heard about it immediately thought of something more to do," says Birman. They started to look for a general expression that encompasses both the Jones and the Alexander polynomials. Success took the shape of a polynomial using three variables and various powers and coefficients to express a knot's properties — and five papers describing the result.



$P_L = yz^{-1} + x^{-1}y^2z^{-1} - x^{-1}z$,

$P_L = x^{-2}z^2 - 2x^{-1}y - x^{-2}y^2$,

$P_L = y^{-2}z^2 - 2xy^{-1} - x^{-2}y^2$,

$P_L = x^{-1}y^{-1}z^2 - xy^{-1} - x^{-1}y - 1$.

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A new polynomial, P_L, that encodes certain properties of various knots and links can distinguish mirror images, as shown for the left- and right-handed trefoils listed in the second and third lines.

In the April Bulletin of the American Mathematical Society, the editor notes: "It was evident from the circumstances that the four groups arrived at their results completely independently of each other, although all were inspired by the work of Jones. The degree of simultaneity was such that, by common consent, it was unproductive to try to assess priority."

In the end, one paper with six listed authors was published. A mathematician not directly involved wrote an introduction, and the four groups presented summaries of their proofs. The fifth group, a pair of Polish mathematicians, was the victim of slow mail and just missed being included in the joint paper.

"I felt very proud of mathematicians for the nice way that those competing announcements were handled," says Birman. "It had the potential for a big argument, but there was none."

The excitement hasn't died down. The new polynomial has prompted all kinds of mathematical questions. Both the Polish mathematicians and Kenneth C. Millett of the University of California at Santa Barbara, also one of the co-discoverers, and his colleagues have found several more independent polynomials that describe aspects of knots.

"So now there are more polynomials than you can count," sighs Birman. "It seems clear that they're all part of a still bigger picture that we don't know yet." But the results are encouraging too. Eventually, a complete invariant that distinguishes any two knots may be found.

Moreover, no one really understands what the new polynomials mean. "There's a procedure for computing this polynomial, and there are all these different proofs that it really is attached to a knot type, but nobody understands geometrically what it means," says Birman.

The polynomial apparently encodes many kinds of existing data about knots but in very strange ways. Furthermore, "it seems quite amazing," says Birman, "that any one polynomial should detect so many different things."

The procedure for finding the polynomial in a given case is also very simple and easy to implement in a computer program, says Jones. The difficulty is that the time needed to compute a polynomial goes up exponentially with the number of "crossings." This makes a knot with, say, 40 crossings almost impossible to check by computer. Whether a faster algorithm exists is still an open question, Jones says.

"The new invariant is simple and powerful, and it is surprising that it has eluded topologists for so long," says Ian Stewart of the University of Warwick in England, commenting in the Sept. 26 Nature. "And mathematicians can take heart from this discovery — not every new idea need be more complicated than old ones." □