Games Mathematicians Play

Mathematical puzzles, games and other pastimes both enlighten and amuse

By IVARS PETERSON

Have you heard the one about an itinerant entertainer traveling with a wolf, a goat and a basket of cabbages? The showman comes to a river and finds a small boat that holds only him and one passenger. For obvious reasons, he can't leave the wolf alone with the goat or the goat with the cabbages. How does he get his cargo safely to the other side?

This well-known brainteaser has been around for centuries. In a version initiated in the 13th century, the puzzle involved three handsome young men who have three beautiful ladies for wives. All six are jealous of their spouses. Using a two-person boat, how many trips does it take to ferry them all across a river — without igniting a fit of illicit passion?

Both of these problems and many other variations on this theme are simply ways of dressing up a relatively straightforward mathematical problem. Since the days of the ancient Egyptians and Babylonians, such devices have often been used to turn a routine mathematical exercise into something that tickles and challenges the mind.

Mathematical puzzles and games are still remarkably popular. Throughout the world, puzzle addicts snap up many of the hundreds of such books published every year. Numerous magazines feature puzzle columns. Furthermore, the appearance of a new and ingenious puzzle can stir up frenzied activity. In three years, for instance, sales of Rubik's cubes exceeded 100 million.

Amusement is one of humankind's strongest motivating forces, says mathematician David Singmaster of the Polytechnic of the South Bank in London, England. Recreational problems, he adds, have spawned many mathematical fields. The origin of probability theory in questions about gambling is just one example.

Moreover, says Singmaster, an interesting problem is worth hours of lecturing. "Recreational mathematics offers a range of problems [that] have fascinated students for generations and should continue to do so for future generations."

Singmaster was one of about 100 people, a mixture of professional and amateur mathematicians, who recently spent a merry week at a conference on recreational and intuitive mathematics at the University of Calgary in Alberta. At this meeting, work was play and play was work. Puns punctuated lectures. Pen or pencil scribbles marked progress toward new puzzles or novel solutions for old ones. Participants wrestled with numbers, tiles, wooden blocks, counters, matchsticks, coins, cards and fiendishly interlocked wire rings.

Even cookies made an appearance. They play a tantalizing role in a new game introduced by mathematician James Propp of the University of California at Berkeley. He describes the game as follows: Imagine two children who take turns stealing cookies from a larder, each taking a single cookie every other day. Some of the cookies may go bad while sitting on the shelf, but fortunately each cookie is frosted with its own expiration date. Once that date is reached, the children avoid the spoiled cookie.

"The goal of each of these mean-spirited children," says Propp, "is . . . to have the spiteful pleasure of getting the last cookie." If no cookies spoil during the game, the game turns out to be very boring. The winner's identity then depends only on whether the number of cookies is odd or even. But when some cookies go bad, the game's outcome is much less predictable.

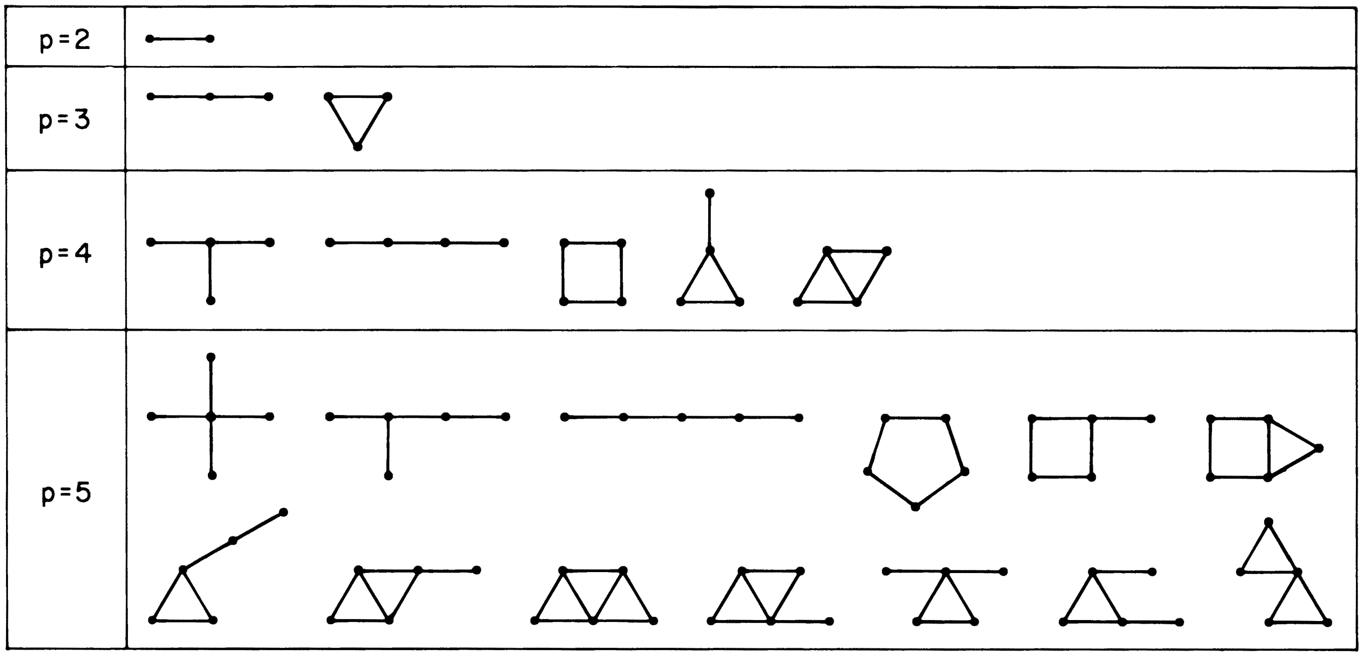
Propp taught his brother how to play the game and promptly lost to him. "He played randomly and beat me," Propp says. Now Propp is trying to analyze the game to see if there are strategies that guarantee a win.

In his analysis, each cookie is represented by a pile of counters, where a pile's height equals the number of days remaining in the cookie's life. On any given turn, the player takes away one of the piles, and one counter is removed from all of the remaining piles. The player who takes away the last pile is the winner.

Although Propp has uncovered some interesting patterns that apply to certain sets of counters, he has not yet found a general strategy that can be used for any starting group of cookies. "I have more than enough data," he says, "but a lack of conjectures."

Heiko Harborth of the Technische Universität Braunschweig in West Germany plays with matches. "Matchsticks are among the cheapest and simplest objects for puzzles," he says. "Whole books have been devoted to matchstick puzzles." At the meeting, to keep his audience fully occupied, Harborth handed out boxes of matches to anyone who preferred working on a puzzle to listening to his lecture.

One group of matchstick (or toothpick) problems involves constructing patterns in which a given number of



Illustrations: Arnis Peterson

Figure 1: *There's only one way in which a matchstick can join two points, two ways for matchsticks to connect three points, and five ways to connect four points. As the number of points increases, the number of possible matchstick patterns escalates rapidly. Can you find the 50 patterns that join six points?*

sticks meet end to end (without crossing each other) at every point in a geometric figure on a flat surface. For example, a figure made up of three sticks laid out as an equal-sided triangle has two sticks meeting at each corner. This is the smallest number of sticks that can be used to create a pattern in which two sticks meet at every vertex.

The problem is tougher when three sticks must meet at every corner. The answer requires a figure made up of a minimum of 12 sticks that meet, three at a time, at eight vertices. Can you find it?

The answer for four sticks meeting at each vertex isn't known. So far, Harborth has found an arrangement made up of 104 matchsticks meeting at 52 points (see cover), but he doesn't know whether this is the smallest construction that meets the criteria. It is known, however, that no such patterns exist for five or more sticks meeting at each vertex.

Playing with matchsticks raises questions that come up in the mathematical field of graph theory — the study of ways in which points can be connected. Such graphs often play important roles in circuit and network design.

Some types of graphs can be explored using matchsticks. For example, there is only one way to use a single matchstick to connect two points. There are two ways, using two or three matchsticks, to connect three points: The points fall in a line or a triangle. There are five different matchstick arrangements that connect four points, 13 arrangements for connecting five points and 50 for connecting six points (see figure 1). But a general formula for determining the number of possible arrangements, given any number of points, hasn't been found.

There are many similar, unsolved problems, says Harborth.

Magic squares have fascinated people for thousands of years. These objects consist of a set of whole numbers arranged in a square so that the sum of the numbers in every row, in every column and along each diagonal is the same. Some magic squares have special properties. In ancient China, for example, a three-by-three square that uses all of the digits from one to nine was said to bring good luck.

Recently, the deciphering of a 5th-century inscription written in the runic alphabet led to the discovery of a new type of magic square. It all began with a rare, 19th-century book called *The Origin of Tree Worship,* says Lee C.F. Sallows of Nijmegen in The Netherlands. This book describes the practices of ancient Celtic priests known as druids and contains a runic charm that was supposed to have

A math library for fun

A puzzle and games enthusiast could spend hours browsing through a special collection of books and other items at the University of Calgary library in Alberta. Devoted to recreational mathematics and related pursuits, this collection may eventually become the world's definitive source of information on these topics.

University of Calgary Libraries

The core of the present collection is a set of 2,000 items that had been put together by Eugene Strens, a Dutch engineer, amateur mathematician, friend of artist M.C. Escher and avid collector. After Strens died in 1980, Calgary mathematician Richard K. Guy was instrumental in bringing the complete collection to the university. A fund for enlarging this unique math library and a system for receiving donations have now been instituted to keep the collection growing.

"The vast amount of material here is really quite delightful," says David Singmaster of the Polytechnic of the South Bank in London, England. Singmaster, who is compiling a comprehensive list of sources in recreational mathematics, spent several weeks going through the Strens collection.

Although the library contains a number of rare books and many of the major works in recreational mathematics, one of the collection's strengths is all the minor items it contains. Most libraries wouldn't consider acquiring many of the cheaply published puzzle books and pamphlets that flood the popular market and then quickly disappear. Yet these books are useful in filling gaps between the landmark volumes.

— I. Peterson

5	22	18
28	15	2
12	8	25

→

five	twenty-two	eighteen
twenty-eight	fifteen	two
twelve	eight	twenty-five

→

4	9	8
11	7	3
6	5	10

44	61	57
67	54	41
51	47	64

→

9	8	10
10	9	8
8	10	9

→

4	5	3
3	4	5
5	3	4

Figure 2: *The deciphering of this 5th-century runic inscription recently led to the discovery of a new class of magic squares. The inscription consists of a set of numbers that fill the spaces in a three-by-three square. Each column, row and diagonal adds up to the same number (45). Remarkably, when the number in each space is replaced by the number of letters in the word for the number, a new magic square is created—one that uses all the numbers from three to 11. It works in both the original language and in modern English. The second row illustrates a newly discovered sequence of magic squares in which numbers have twice been replaced by the number of letters in their names to produce three related, albeit language-dependent, magic squares.*

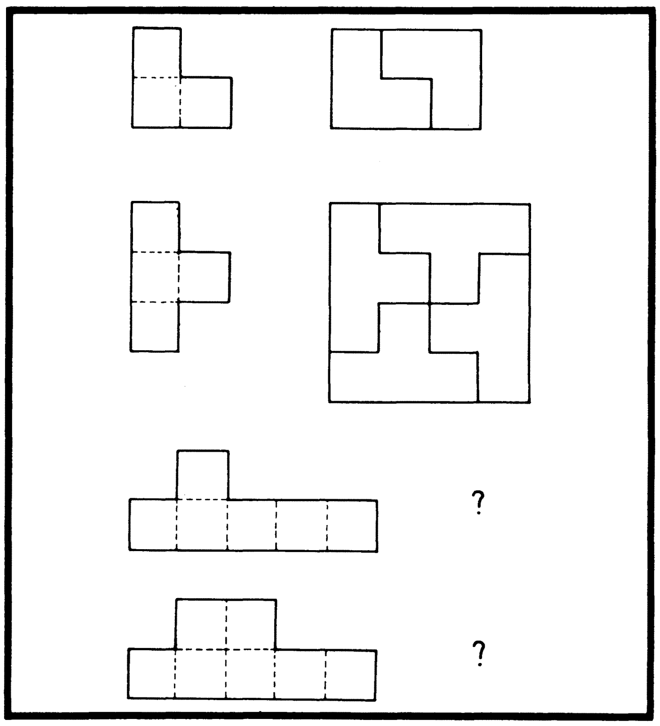
magical properties. Sallows helped translate the charm into modern English.

The translation revealed that the charm is actually a three-by-three magic square. But this one has a remarkable property. When the number of runic characters that make up each number in the original magic square is written in the appropriate space, the derived numbers also form a magic square! Moreover, this second magic square is made up of consecutive integers from three to 11. Curiously, the same pattern also works in modern English (see figure 2).

It was an astonishing discovery, says Sallows.

This led to a search for other examples that have the same property, not only in English but in many other languages. Sallows found that for columns, rows and diagonals totaling less than 200, French has only one such magic square while English has more than seven. Welsh, on the other hand, has more than 26. For totals less than 100, he found none in Danish, six in Dutch, 13 in Finnish and an incredible 221 in German. He even discovered a three-by-three English square from

Figure 3: *One set of tiling problems involves showing whether any number of tiles of a given shape can be fitted together to form a rectangle. Such rectangular arrangements have been found for many tile shapes, but not for the two pieces shown. At the same time, no one has been able to prove that it can't be done.*



which a magic square can be derived, which in turn yields another magic square.

The search has now expanded to four-by-four and five-by-five language-dependent magic squares. Sallows describes his quest as "a search for ever more potent magic spells." An account of his peregrinations is scheduled to appear in ABACUS.

Tiling problems also have a long history. Evidence that many have been solved, often in ingenious ways, is seen in the folk art of many cultures, in the intricate Moorish mosaic patterns found at palaces like the Alhambra in Spain, and in the works of 20th-century Dutch artist M.C. Escher. But there are numerous tiling questions that haven't been settled yet, says Solomon K. Golomb of the University of Southern California in Los Angeles.

One problem involves using pieces of a given shape to form a rectangle (see figure 3). With square and rectangular pieces, the problem is easy to solve. But a piece made up of six squares, connected so that five lie in one row with a sixth attached to the row's second square, is much more difficult to deal with. In fact, no one has yet been able to prove that tiles of this shape can or cannot be laid down to form a full rectangle.

Settling the question one way or the other would be worth a significant prize, says Golomb. Also unknown is whether a certain tile made up of seven squares can be used to form a rectangle.

Another tiling question is related to the recent discovery of quasicrystals, which appear to have a nonperiodic crystal lattice (SN:3/23/85,p.188). Years before, Roger Penrose, a physicist at Oxford University in England, had discovered a tiling pattern consisting of diamond-shaped tiles, some fat and some thin, that fitted together to create a pattern that did not repeat itself at regular intervals.

The open question, says Golomb, is whether a similar nonperiodic tiling can be built up from a single type of piece rather than from the two different types of pieces found in all known examples. "Is there a single tile that will do it in a nonperiodic way?" he asks.

Other geometric recreations discussed at the meeting included the art of paper folding (origami) to create three-dimensional geometric shapes. Thomas Ritchford of Brooklyn, N.Y., handed out instructions for constructing complex star-like figures from paper without using scissors or other mechanical aids. Made up of simple modules fastened together in ingenious ways, the completed models hold together without glue. The constructions also provide lessons in the geometry of identical pieces that fit together to form a cube.

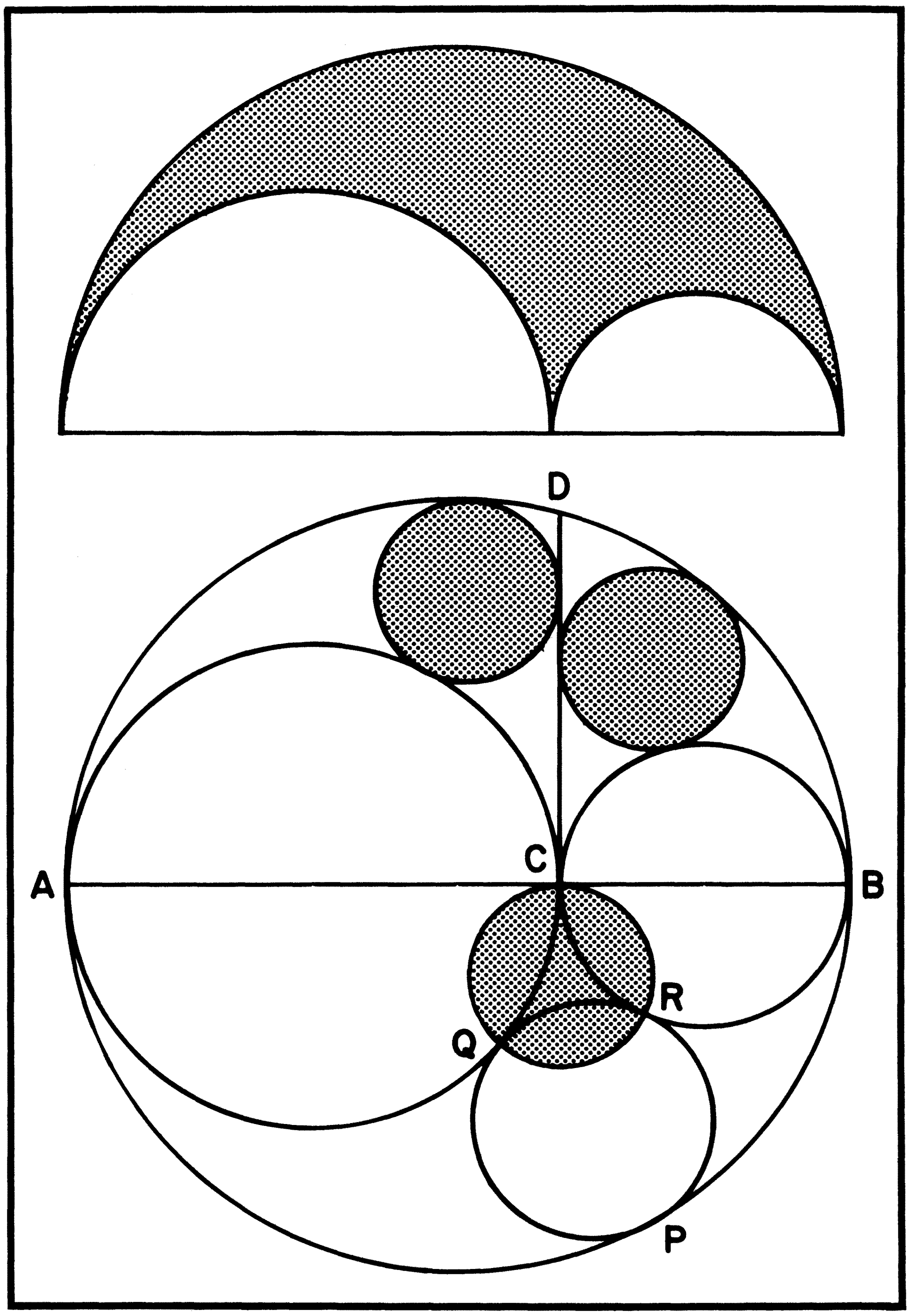


***Figure 4:** The "shoemaker's knife" or "arbelos" (shaded area in top diagram), first described centuries ago by Archimedes, has a host of striking geometric properties, many of which have gone unnoticed for years. Amateur mathematician Leon Bankoff, for example, discovered a third circle (CQR) equal in size to the long-known twin circles shown on either side of the line DC.*

Dentist Leon Bankoff of Los Angeles described the results of many years spent exploring the "intuition-shattering" geometric properties of a figure called the "shoemaker's knife" (see figure 4). This geometric figure, first described by Archimedes and known to the ancient Greeks as the arbelos, has been a rich source of curious and unexpected discoveries for centuries.

One thing that the meeting made clear was that nonmathematicians can contribute to and participate in mathematics research. Not all mathematical research topics dip into arcane notation and mind-numbing mental gymnastics.

Recreational mathematics, says Singmaster, is something that a lot of people can do. The tools are simple — often just pencil and paper. All it takes is a lot of patience and persistence. That's something that all of the conference participants seemed to have in common.

But beware. Once a puzzle takes hold, it demands attention. Daniel Ullman of George Washington University in Washington, D.C., remarks on one problem he wrestled with recently: "I wasted most of January, February and March on it."

That's one of the surest signs that a recreational mathematician is at work. □