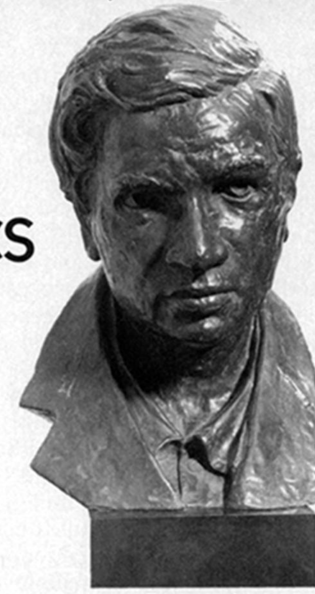
The Formula Man

The legacy of India's greatest mathematician continues to influence modern mathematics



Sculpture: Paul Granlund/Photo: Bill Fritsch

Ramanujan

By IVARS PETERSON

Fragments of sunlight pattern a frail figure. He scribbles on a slate – scratching, erasing, pausing, then rapidly putting down a chain of digits and algebraic symbols. He smiles briefly. He reaches for his notebook and carefully enters the final result: a new formula relating two different mathematical entities. There's no formal proof; paper is too precious for such details. He goes back to his slate to experiment and to test his ideas further.

Although this imagined picture suggests how it may have happened, no one really knows how Srinivasa Ramanujan came up with his astonishing array of mathematical discoveries. Born in 1887 in southern India, Ramanujan lived most of his life in obscurity and poverty. Largely self-educated and hooked on mathematics, between 1903 and 1914 he filled three notebooks with page after page of mathematical formulas and relationships – perhaps as many as 4,000 theorems, all stated without rigorous proof.

"We have no idea how he did the marvelous things he did, what led him to them, or anything else," says mathematician Richard A. Askey of the University of Wisconsin in Madison. "But he's an inspiration to many of us."

Although Ramanujan died in 1920 at the age of 32, his work is still the subject of considerable interest. Recent results such as the computation of pi to millions of digits (SN: 2/21/87, p.118) and the solution of a particular mathematical model in two-dimensional statistical mechanics rely completely or in part on Ramanujan's pioneering research. Moreover, several mathematicians have spent the last decade or so going through Ramanujan's notebooks, systematically proving each of his theorems. This year marks the hundredth year since Ramanujan's birth, and several conferences highlighting his accomplishments and his impact on modern mathematics are scheduled to take place in the United States and India.

Ramanujan's life as a professional mathematician really began in 1914, when he accepted an invitation from the prominent mathematician G.H. Hardy to come to Cambridge University in England. The year before, Ramanujan had sent samples of his work to several people in England, including Hardy, and Hardy had been impressed. Even without proofs, the results had to be true, Hardy said, because no one would have had the imagination to dream them up.

Ramanujan spent five years in England, publishing a considerable number of papers and achieving worldwide fame. In 1917, he contracted a mysterious, incurable disease. He spent the last year of his life back in India.

Speaking about that final year, Ramanujan's widow, still alive today, has said: "He lived for less than a year. He was only skin and bones. He often complained of severe pain. In spite of it, he was always busy doing his mathematics. That, evidently, helped him to forget his pain. I used to gather the sheets of paper which he filled up. I would also give the slate whenever he asked for it."

Much recent effort has gone into editing Ramanujan's notebooks and the 130 pages of scribbled material (now known as the "lost" notebook) from the last year of his life. "I employ the word 'edit' in a broader sense than is normally used," says Bruce C. Berndt of the University of Illinois at Urbana-Champaign, a mathematician who has spent much of the last decade working on Ramanujan's notebooks. "If a formula or theorem is considered to be new, I attempt to prove it. If a formula or theorem is thought to be known, I find sources for it, but the great majority of my time is spent in trying to prove the hitherto unproven results found in the notebooks." Even after more than 65 years, much of what Ramanujan accomplished has not been rediscovered or duplicated.

"Since the notebooks were intended for [Ramanujan's] own use," says Berndt, "it is amazing that they contain very few serious errors." After working through the notebooks, he says, "I still don't understand it all. I may be able to prove it, but I don't know where it comes from and where it fits into the rest of mathematics."

Berndt

A page from Ramanujan's second notebook.

Even the 600 or so results from the last year of Ramanujan's life contain intriguing assertions that are not yet proved. George E. Andrews of Pennsylvania State University in University Park, who a decade ago discovered the pages of Ramanujan's nearly illegible, unlabeled jottings stored away at Cambridge University's Trinity College library, has managed to decipher and prove many of the manuscript's theorems. "He was able to pick out and touch upon functions of great interest," says Andrews. "As we study them, we realize how revolutionary and how intriguing they are, and the real implications they have for mathematics in the current day."

"The topic that Ramanujan loved the most is infinite series," says Berndt. "They are copiously found throughout the notebooks." Mathematically, a series is simply the sum of a specified number of terms. Usually, a simple algebraic formula indicates what each successive term should be. For example, the series 1 + 4 + 9 + 16 can also be written as the sum of all terms n^2, when n runs from 1 to 4. In the case of an infinite series, the summation process continues forever.

One of Ramanujan's infinite series is now the basis for methods used to compute pi (π) to millions of decimal places:

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{[1103 + 26390n]}{396^{4n}}$$

In this case, the Greek letter sigma (Σ) signifies summation, and the numbers above and below the symbol indicate that the formula is exactly correct only if all terms, starting with $n = 0$ and going to infinity, are added together. An exclamation mark, rather than expressing excitement, means that the designated number is actually the product of all numbers up to and including the number shown (that is, 4! would be $4 \times 3 \times 2 \times 1 = 24$).

This equation, says Peter B. Borwein of Dalhousie University in Halifax, Nova Scotia, "is the most rapidly converging sum for pi that I know." Each extra term in the summation adds roughly eight digits to the decimal expansion of pi. Bill Gosper of Symbolics Inc. in Palo Alto, Calif., used this formula as the basis for his 1985 calculation of pi to 17.5 million digits (SN: 2/8/86, p.91). At that time, no proof that the sum was actually true existed.

In one of his papers, Ramanujan gives 14 other series for $1/\pi$ and, as usual, offers little explanation of where they came from. Says Borwein, "It's not possible to deal with Ramanujan's work without being awed by what he managed to accomplish." Even now, with the aid of a more profound theoretical understanding and helped by new mathematical tools such as computer software for manipulating algebraic expressions, says Borwein, "it's still very hard to generate the kind of identities that Ramanujan was finding."

Ramanujan also expended a great deal of effort inventing and exploring structures called continued fractions. "In this area," says Berndt, "Ramanujan is probably unsurpassed in all of mathematical history."

In a sense, continued fractions, as the name hints, are fractions of fractions of fractions of fractions. . . . The following expression for pi is a typical example of one of these typographical nightmares:

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cfrac{9^2}{2 + \ddots}}}}}}$$

Each successive term in the continued fraction expansion sits deeper and deeper within the fraction's denominator. As in the case of a given series, a simple algebraic formula can be used to generate these terms.

"Perhaps his most interesting and enigmatic continued fractions are for various products and quotients of gamma functions," says Berndt. Gamma functions involve the values of certain integrals in calculus. "We do not know how he made these discoveries," he says, "nor do we fully understand this particular topic."

In a way, Ramanujan's fascination with formulas was old-fashioned. The great period for the discovery of such relationships was in the 18th and 19th centuries, in the days of Leonhard Euler and Karl Friedrich Gauss. Now mathematicians are beginning to realize that Ramanujan, rather than being a hundred years late, was many decades ahead of his time. Some of his basic theories are just being developed, says Berndt.

"Most mathematicians think of themselves as Platonists," says Askey. "We discover mathematics rather than invent it. Therefore, in the long run, it doesn't really matter what an individual mathematician does. If one person doesn't do it, within 50 years somebody else will probably discover it. The one counterexample to that that I know of in this century is Ramanujan. I'm still constantly amazed at what he did."

Says Borwein, "Had he not died so young, his presence in modern mathematics might be more immediately felt. Had he lived to have access to powerful algebraic manipulation software, such as MACSYMA, who knows how much more spectacular his already astonishing career might have been?" □