TWISTS OF SPACE

Cox, Idaszak & Francis

An artist, a computer programmer and a mathematician work together to visualize exotic geometric forms

By IVARS PETERSON

The Möbius strip is one of the more familiar yet intriguing objects in mathematics. Discovered in 1858 by German astronomer and mathematician August F. Möbius, it can be constructed simply by gluing together the two ends of a long, narrow strip of paper after one end has been given a half twist. The surprising result is a twisted three-dimensional form that has only one side and one edge.

The Möbius strip (or band) is one example of a wide variety of geometrical forms that play important roles in the mathematical field of topology. Topologists emphasize the properties of shapes that remain unchanged, no matter how much the shapes are stretched, twisted or molded so long as they aren't torn or cut. For example, a doughnut and a coffee mug are topologically equivalent. Because both forms have exactly one hole, one can imagine smoothly deforming a doughnut-shaped piece of clay to produce a mug with a single handle.

Like a Möbius strip, a hemispherical bowl has a single edge. If a disk, also having just one edge, is sewn to a hemisphere, the result is a closed, two-sided surface that is topologically equivalent to a sphere. That stitched object can be easily inflated into the shape of a ball.

What happens if a Möbius strip is sewn to the edge of a disk? The possible answers to this question have, over the last century, carried mathematicians deep into the strange and twisted world of four- and higher-dimensional forms. Most recently, this question has formed the basis for a unique collaboration involving an artist, a computer programmer and a mathematician working at the National Center for Supercomputing Applications in Urbana, Ill.

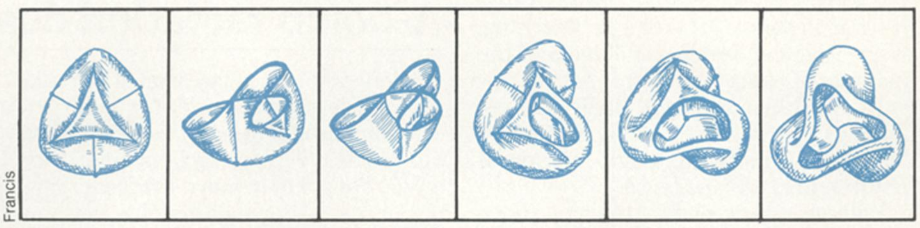
One of the earliest answers to the Möbius-strip stitching problem was found among geometrical constructions made by German mathematician Jacob Steiner during the early part of the 19th century. His "Roman surface," named in memory of a particularly productive stay in Rome, fits the requirements of a Möbius strip sewn to a disk. Steiner's Roman surface, at least from one viewpoint, looks like a severely deformed bowl with a fat, pinched lip.

In 1900, Werner Boy discovered a "simpler" surface that also meets the same criteria. His convoluted surface looks like a pretzel that has stepped out of "The Twilight Zone." However, Boy couldn't find the algebraic equations that would specify the location of every point defining its shape. All he could do was to describe cross sections through the surface. That was enough to construct a wire framework or sculpt a plaster model but not enough to find a formula for the surface. French mathematician François Apery, a student of topologist Bernard Morin at the University of Strasbourg, finally worked out the equations in 1984.

Because both the Roman surface and the Boy surface are closely related, it was natural for mathematicians to look for orderly ways of transforming one surface into the other. Such a procedure, known as a homotopy, involves a careful cancellation of singularities — places where the surface twists into itself to form a double curve or where the surface is pinched and abruptly changes direction.

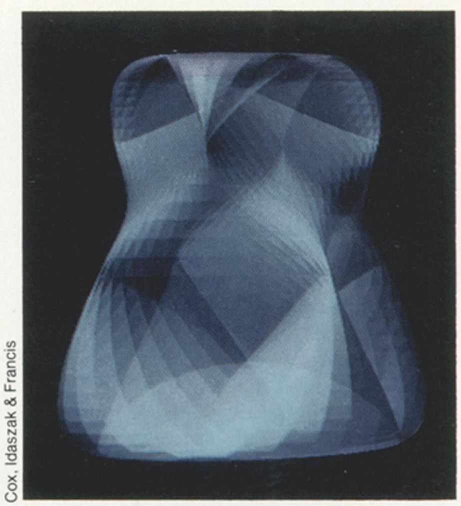
Singularities can often be removed by thinking of the figure as a higher-dimensional form. For example, the two-dimensional shadow of a bent wire loop sometimes appears to cross itself, although the three-dimensional loop itself doesn't actually intersect anywhere. Likewise, what appears in three dimensions to be a pinch point could very well be a perfectly regular feature in four dimensions. The pinched three-dimensional form is merely a particular shadow cast by the four-dimensional surface.

The Romboy homotopy — the name for the transformation from the Roman to the Boy surface — is based on the idea that both the Roman and Boy surfaces can be generated by moving an oval or ellipse through space. A circle, for example, when rotated through 180 degrees, gener-



Francis

The topological transition from Steiner's Roman surface (far left) to Boy's surface (far right) is based on the cancellation of pinch points. The second and third diagrams in the sequence show cross sections of the Roman surface as one pinch point is cancelled out. Further alterations lead to Boy's surface, sliced open to provide a clearer picture of its complex shape.



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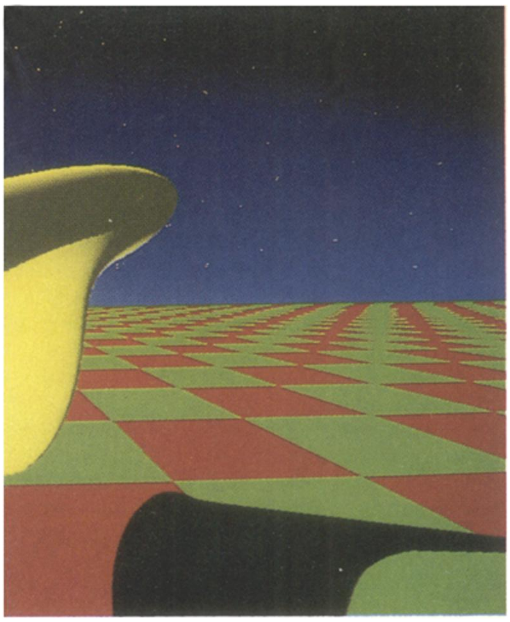
This transparent, computer-generated image of the topological form known as the Etruscan Venus allows a viewer to peer into the figure's convoluted interior.



A computer-generated incarnation of Boy's surface, discovered by mathematician Werner Boy in 1900, casts a shadow across a surrealistic landscape.

ates a sphere. In the same way, an ellipse, its motion governed by well-defined constraints, can stretch and contract as it sweeps out a wobbly path to produce a particular surface. This bouquet of ellipses defines the shape.

Apery, starting with the Roman surface, discovered that by smoothly altering the parameters governing the requisite motion of an ellipse, he could gradually transform the Roman surface into the Boy surface. In fact, the equations governing this procedure show that both surfaces can be considered three-dimensional "shadows" of a higher-dimensional form viewed from different vantage points.

The existence of Apery's equations made it possible to program a computer to display the two figures and the homotopy linking them. Mathematician George K. Francis of the University of Illinois at Urbana-Champaign, who has long been interested in methods for visualizing geometrical forms, programmed an Apple computer to generate a rough version of the homotopy. Such computer sketches are the equivalent of the wire frameworks and hand drawings frequently employed by mathematicians a century ago to study mathematical forms.

When artist Donna J. Cox, who was working with computer programmer Ray Idaszak, came to Francis looking for a mathematically oriented project to do at the university's new supercomputing center, Francis suggested they try recreating the Romboy homotopy. Francis, Cox and Idaszak became what Cox terms a "Renaissance" team, with each person making important contributions to the project.

It took less than two weeks for the group to convert the original Apple program into one usable on a supercomputer. They added shading and color to the computer drawings, putting a "skin" on what originally were little more than skeletons of the surfaces. The result was a short, 600-frame, computer-drawn film that shows a smooth deformation from the Roman to the Boy surface, then back to the original. "This deformation had never been realized before as an animation," says Cox.

In the course of fiddling with the computer program, Francis also discovered a simpler way of performing the homotopy. Instead of generating the surfaces by letting a wobbling ellipse sweep through space the whole time, Francis also used a mathematical figure known as a limaçon. This figure looks like a closed loop coiled so that it crosses itself once to form a double loop. It can also take on a roughly heart-shaped form.

By following this alternative homotopy, the three researchers stumbled upon a new surface that appears along the way. From one point of view it looks somewhat like an owl, and from another, like a female torso. The new surface was dubbed the "Etruscan Venus." The term Etruscan reflects the idea that the homotopy used is simpler or more primitive than the original Apery homotopy, just as the ancient Etruscan civilization in Italy predated the Roman Empire.

The Etruscan Venus turns out to be a form known as a topological Klein bottle. It can be created by gluing together two Möbius strips along their edges, something that can be done only in the roominess of four dimensions. Just as a disk is the shadow of a three-dimensional sphere, the Etruscan Venus is the three-dimensional shadow of a four-dimensional Klein bottle.

Francis and his team also applied his version of the Romboy homotopy to the Boy surface generated by Apery's homotopy. Again, they discovered a new mathematical shape. Because Idaszak was the first to see it, this one came to be called "Ida." Like the Etruscan Venus, Ida is also the shadow of a four-dimensional Klein bottle.

All three participants got something worthwhile out of the project. Cox now has a kind of sculpture machine — an interactive computer program, strongly rooted in the mathematics of topology, which is proving to be a rich source of dramatic and beautiful shapes. By manipulating the 10 parameters (or dimensions) in the equations defining the Romboy homotopy, she can create an endless array of entirely new forms. "It's fascinating to watch and work with on the screen," she says. "You can tug on one dimension and transform things radically. You can see surfaces that are really quite surprising."

"The artistic aspect [of the project] now has a life of its own," says Francis. "By manipulating formulas, entirely new shapes that have never been seen before can be created."

Recently, Cox has been working with two artists in Chicago to create special three-dimensional images of some of these newly discovered surfaces. Several will be on display later this fall at Chicago's Museum of Science and Industry.

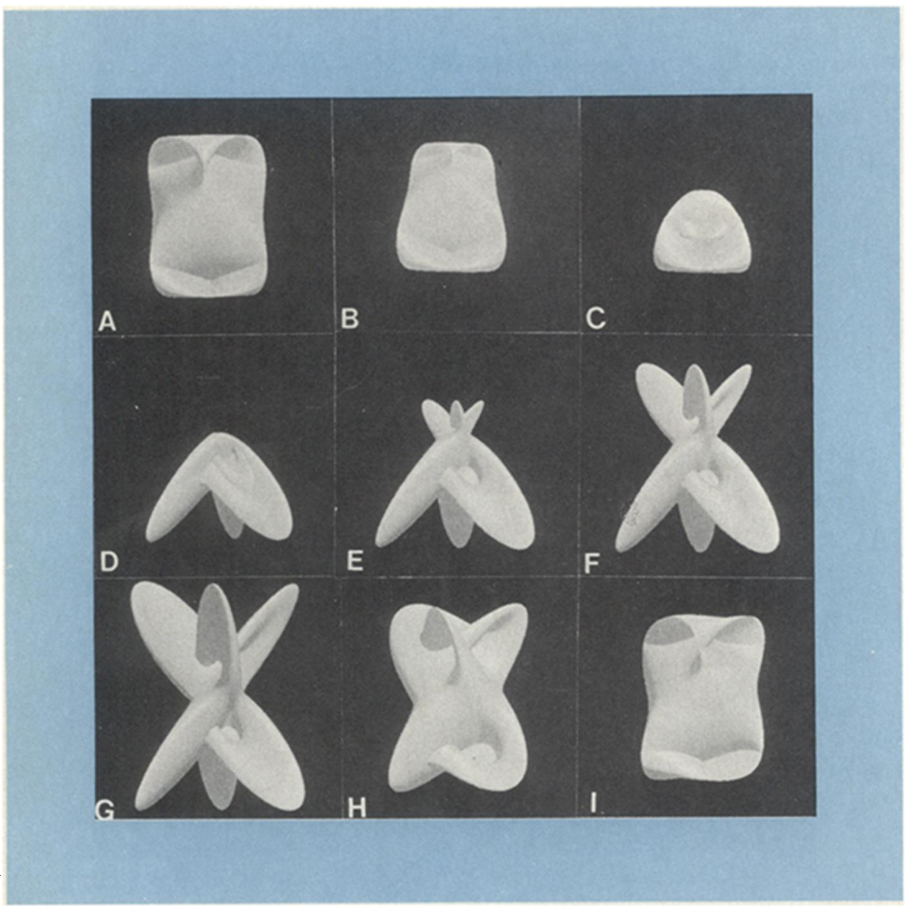
abruptly reverse their direction or penetrate themselves. Such features often confuse conventional computer programs designed for coloring in and shading geometric figures. Idaszak's methods work equally well on a microcomputer and a supercomputer. The main difference is a matter of scale.

For Francis, the project has generated some new mathematics and effectively demonstrated the important role that computer graphics can play in mathematical research. Inspired by Morin, who, though blind, has been one of the strongest proponents of visualization in topology, Francis has spent much of his career promoting the use of drawings and models to aid in formulating definitions and proofs in mathematics.

In his new book, "A Topological Picturebook" (Springer-Verlag, 1987), Francis writes: "It is . . . in the making of the model, in the act of drawing a recognizable picture of it, or nowadays, of programming some interactive graphics on a microcomputer, that real spatial understanding comes about. It does this by showing how the model is generated by simpler, more familiar objects, for example, how curves generate surfaces."

Francis is particularly interested in developing computer tools that make it easy and quick for students and mathematicians to "sketch" geometrical figures and explore their properties. "We're talking about computer graphics, first of all, as a way of illustrating theorems that are already known, and secondly, as an experimental tool for finding new conjectures," he says. What's needed is "an inexpensive graphics tool for producing recognizable images, in huge quantities and fast enough to keep from wasting time." The first steps toward that goal have already been taken.

The Illinois collaboration showed how well a disparate group of people can work together. "The project worked better than we had any reason to believe," says Francis. The team is now interested in producing a definitive version of the Romboy homotopy, seen from angles useful to mathematicians and with careful use of color and shading to illuminate important features.



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This selection of frames from the videotape "Metamorphosis: Shadows From Higher Dimensions" illustrates steps in the Romboy homotopy. The Etruscan Venus (A) is transformed first into the Roman surface (C). Next, the Roman surface is turned into the Boy surface (D), which in turn evolves into a new surface called Ida (G). Finally, Ida turns into the Etruscan Venus (I or A), completing the cycle.

Cox is also interested in exploring how artists can help researchers find better ways of conveying the information represented by vast amounts of data. Her studies of the role of color have already helped an astronomer and an entomologist (SN:1/10/87,p.20) present their results graphically. "I'm interested in using color to get more information out of a system," says Cox, "for demystifying and clarifying images." Pictures of "Ida," for example, have been colored to indicate the orientation of the ovals used to create the surface.

Meanwhile, Idaszak has developed new computer graphics techniques that make it easier to picture surfaces that

"The common thread throughout this collaborative research has been the use of advanced technology and color techniques to make visible the multidimensional layers of abstract information," writes Cox in an article to appear in LEONARDO. "Supercomputers, graphics, and creative human beings have the power to bring about visual enlightenment with regard to much in this universe that was formerly abstruse mathematics." □