A Shortage of Small Numbers

Mathematician Richard K. Guy of the University of Calgary in Alberta is a collector. He patiently and painstakingly searches far and wide for the unexpected and the quirky among the family of whole numbers. He looks for unusual patterns.

Identifying patterns and asking the right questions are two of the most important ingredients of mathematical research. Lamentably, there's no foolproof recipe for generating good questions and no formula for recognizing whether an observed pattern will lead to a significant new theorem or is merely a lucky coincidence. Until a mathematical proof is constructed to settle the question, a mathematician must rely on fallible, empirical evidence.

Consider the remarkable sequence of integers 31, 331, 3331, 33331, 333331, 3333331. Each of these is a prime number, that is, divisible only by itself and the number one. Is the sequence's next number, 33333331, a prime? The answer is yes. Sadly, the pattern falls apart with the succeeding number, 333333331, which turns out to be the product of 17 and 19,607,843. A promising pattern is slain by a cruel counterexample.

Guy's specimens are all instances of sequences that depend on the value, *n*, of some parameter. In the first example, *n* represents the number of threes in each integer. The pattern works for $n = 1, 2, 3, 4, 5, 6$ and 7, but fails when $n = 8$. For any sequence that depends on the value of *n*, experience shows that sometimes a pattern persists but frustratingly often the pattern is simply a figment of the smallness of the values of *n* for which the example has been worked out.

For many years, Guy has been trying to encapsulate his findings in the form of a universal law. So far, the best he can manage is the statement: "There aren't enough small numbers to meet the many demands made of them." He calls it the Strong Law of Small Numbers.

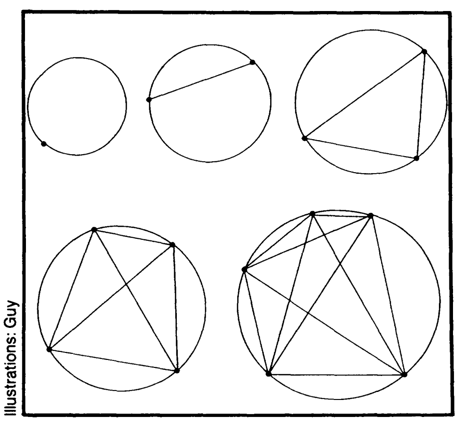
"It is the enemy of mathematical discovery," Guy says. "When you notice a mathematical pattern, how do you know it's for real? We are easily led astray by spurious patterns, which do not continue as the numbers get larger. On the other hand, genuine patterns are often hidden by a few exceptions near the beginning."

As an instance of the misleading behavior of small numbers, Guy cites the fact that 10 percent of the first 100 numbers are perfect squares (1, 4, 9, 16, 25, 36, 49, 64, 81 and 100). On the basis of this pattern, one could conjecture that 10 percent of the first 1,000 numbers would also be perfect squares, but some quick calculations show the conjecture is ill-founded.

On the other hand, the statement that all prime numbers are odd is almost true. The only exception occurs right at the beginning. In a sense, as Guy points out, two is the "oddest" prime.

Guy's tussles with such aberrant numerical behavior have led him to formulate an important, elegantly simple theorem: "You can't tell by looking." The theorem, he insists, "has wide application, outside mathematics as well as within," and it can be "proved by intimidation."

Many of Guy's examples, gathered from numerous sources, concern prime numbers. One of the most famous concerns numbers of the form $P = 2^{2^n} + 1$. When $n = 0$, $P = 2^{2^0} + 1 = 2^1 + 1 = 2 + 1 = 3$, a prime number; for $n = 1$, $P = 2^{2^1} + 1 = 5$, another prime; for $n = 2$, $P = 17$; for $n = 3$, $P = 257$; for $n = 4$, $P = 65{,}537$. The numbers 3, 5, 17, 257 and 65,537 are all primes. Does this pattern continue? Mathematician Pierre de Fermat thought so when, centuries ago, he proposed that all numbers of the form $2^{2^n} + 1$ are prime. Alas, when $n = 5$, the answer is not a prime but the product of 641 and 6,700,417. The Strong Law strikes again.

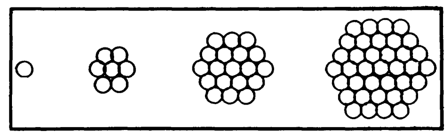


Illustrations: Guy

A tastier problem (diagrammed above) concerns slicing a round cake into pieces — not in the conventional way but in a fashion that probably only a mathematician would appreciate and find edifying. The idea is to define a certain number of points, *n*, along the cake's rim, then to slice the cake so that each cut joins all possible pairs of points. The question is how many separate pieces of cake are created by the cuts.

The answer for $n = 1$ (one point on the rim) is, of course, one. With only one point, no cuts can be made. When $n = 2$, a cut joins two points, dividing the cake into two pieces. For $n = 3$, the number of pieces, *p*, is four; for $n = 4$, $p = 8$; for $n = 5$, $p = 16$. The sequence 1, 2, 4, 8, 16 looks familiar. Does the pattern hold for larger numbers of points? If not, is there another formula that better expresses the observed sequence of numbers?

Pennies show up in a low-budget problem (diagrammed below) of building



hexagons. Seven pennies can be laid out to form a hexagon in which each side is two pennies long. A hexagon with each side made up of three pennies consists of a total of 19 pennies. As the length of the hexagon's side goes from 1 penny to 5 pennies, the total number of pennies involved is 1, 7, 19, 37 and 61. The members of this sequence are called "hex" numbers. Interestingly, $1 + 7 = 8$, $8 + 19 = 27$, $27 + 37 = 64$, $64 + 61 = 125$. Each of these partial sums appears to be a perfect cube. For example, $8 = 2^3 = 2 \times 2 \times 2$, $27 = 3 \times 3 \times 3$, and so on. Does this pattern continue when larger hexagons built from pennies are included?

Guy's collection illustrates the major role that disinformation plays in the pursuit of mathematical truth. He and his fellow collectors could fill many volumes with examples of how the Strong Law of Small Numbers has pointed to significant theorems, has misled investigators into looking for theorems that are not there, or has suggested theorems that may be there but resist all efforts to prove them.

Thirty-five of Guy's favorite specimens are to be displayed in an article that will appear soon in THE AMERICAN MATHEMATICAL MONTHLY. Meanwhile, his collection of numerical curiosities continues to grow. — *Ivars Peterson*