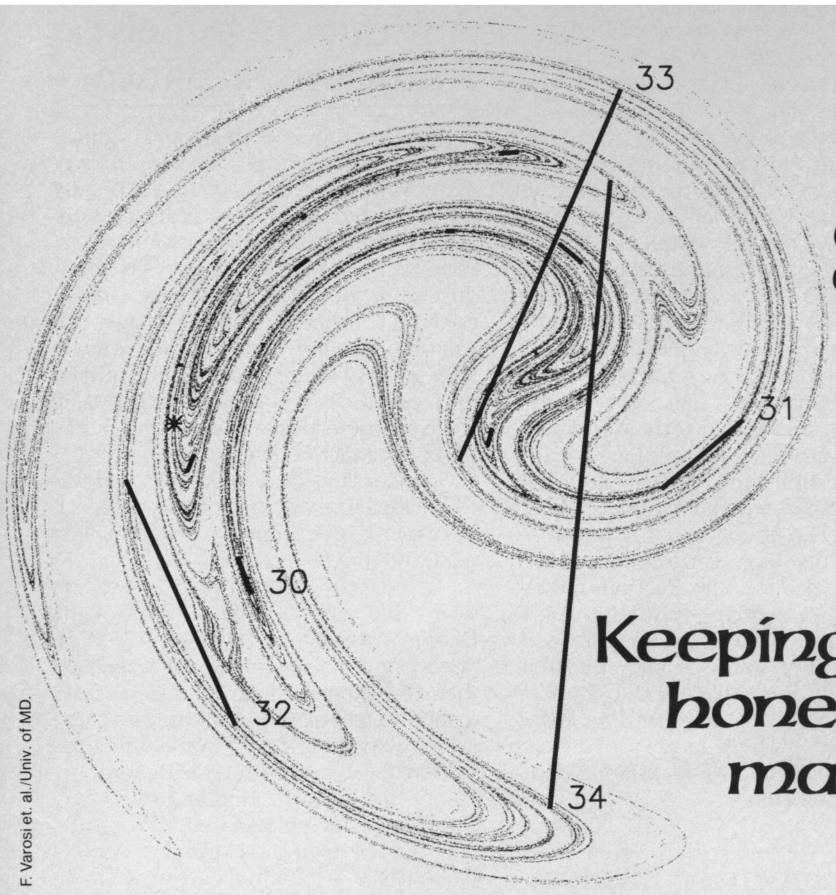

F. Varosi et. al./Univ. of MD.

In the Shadows of Chaos

Keeping chaotic orbits honest takes a keen mathematical eye

By IVARS PETERSON

Like a detective shadowing his errant quarry, mathematician James A. Yorke can track the erratic hops of a chaotic process—carefully checking how closely each step sticks to a true path. Such chaotic processes arise whenever small uncertainties in successive steps in certain repetitive mathematical procedures accumulate so rapidly as to destroy any trace of a pattern.

Yorke, who heads the Institute for Physical Science and Technology at the University of Maryland in College Park, seeks to understand and tame these chaotic processes. His studies start with a repetitive mathematical procedure that, on the surface, appears quite straightforward, even mindless: Calculate the value of a given mathematical expression, or function, for some initial value; then substitute that answer back into the original expression to get a new value, and so on. This simple iterative process often leads to surprisingly complex, unpredictable mathematical behavior.

The question of predictability is significant because the iteration of appropriate mathematical expressions is a standard method for approximately solving the equations used to describe the dynamical behavior of materials, fluids and other physical systems. For example, scientists have developed sets of equations for modeling atmospheric processes to predict changes in weather patterns and climate. Often, they base predictions on the results generated by iterating equations thousands of times. How good can those predictions be if the initial conditions are generally known only to one or two decimal places and the answers coming out of a computer may be intrinsically uncertain? Similar problems come up when researchers compute the way a metal may fracture or how air sweeps past an airplane wing.

"It's worthwhile knowing ahead of time when you can't predict something," Yorke says.

A straightforward calculator experiment illustrates what can happen when iterating even a simple mathematical expression. Substituting the number 0.2 for x in the expression $4x-x^2$ gives the answer 0.64. Continuing that process using successive answers produces a seemingly haphazard sequence of numbers: 0.2, 0.64, 0.922, 0.289, 0.822, 0.586, 0.970, 0.116, 0.406, and so on. Though completely determined by the equation, the sequence jumps around in an apparently random fashion.

Further numerical exploration turns up another surprise. A slightly different starting value leads to a sequence bearing little resemblance to one initiated by a near neighbor. A seed value of 0.21, for instance, produces the sequence of numbers: 0.21, 0.664, 0.892, 0.364, 0.926, 0.274, 0.796, 0.650, 0.910, and so on, a far cry from the sequence starting at 0.2.

The same kind of chaotic behavior turns up in the iteration of many different functions. For certain functions, successive points can be plotted on a graph to produce a two-dimensional cloud of dots. Chaos specialists refer to the sequence of plotted points — one dot leading to the next — as a trajectory, or orbit.

Researchers term such a trajectory chaotic if it jumps erratically from dot to dot, never settling down into any kind of regular pattern. Nevertheless, the motion tends to stay within a bounded region, and some neighborhoods may be visited more often than others. Furthermore, a tiny shift in starting point produces a very different sequence of dots, although the overall dot pattern remains roughly the same.

But computation is intrinsically inexact. If a small change in starting point leads to rapidly diverging results, then errors made when rounding off numbers during a computation may also influence the results. How much does seemingly

In a chaotic process represented by a certain iterated mathematical expression (shown above), two neighboring orbits starting at points differing only in the 14th decimal place (marked by asterisk) will be widely separated (shown by lines) after more than 30 steps, or iterates.

chaotic behavior depend on calculator or computer inexactitude?

For example, consider a computer working with numbers to an accuracy of 14 decimal places. Computer experiments show that two neighboring orbits starting at points differing only in the last decimal place will look totally unrelated after a few dozen steps (see illustration). For some iterated functions, it's not unusual for the distance between orbits to double at every step.

These results imply that a tiny error in rounding off at the first step is sufficient to destroy any attempt at predicting where the orbit is likely to be after, say, 50 iterates. On top of that, errors occur not just in the initial conditions but also at every step.

"This problem faces us because we can do lots of computation," Yorke says.

On the positive side, researchers already have reasons to believe that chaotic orbits are more than just numerical artifacts resulting from computer errors. "You try different computers, you get the same pictures," Yorke says. "You try computing to different [numbers of decimal places], and you still get the same pictures. The macroscopic features stay the same. Only the microscopic features change."

Yorke and his collaborators Stephen M. Hammel and Celso Grebogi have found a way to track the computed sequence of steps, or trajectory, followed by a chaotic process to verify that it stays on a "true" path – a path calculated exactly without any error. They describe their method in the October BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY.

"We have developed rigorous numerical procedures to prove there exists a true orbit that stays near the noisy orbit of a given chaotic process for a long time," the researchers write.

The idea is that while a numerical orbit will diverge rapidly from the true orbit with the same initial point, there often exists a true orbit with a slightly different initial point that stays near or shadows the computed (noisy) orbit dot by dot for a long time – for as many as 10 million steps if computational errors are no larger than the 14th decimal place.

"We're making a rigorous determination of how long a true trajectory stays near a numerical one," Yorke says. That's done by keeping close tabs on round-off errors. The computer does all the necessary arithmetic.

As it calculates a trajectory, the computer places a carefully constructed numerical box, within which a true orbit must lie, around each point. When it proceeds to the next point in the trajectory, it carries the box in a somewhat distorted form along with it. If the original box and the new box overlap in just the right way, then at least one true trajectory stays boxed near the numerical trajectory.

Depending on the computer's numerical precision, the boxing scheme can be extended to at least the first 10 million steps. Such long shadowing times are striking when compared to the great rate at which orbits diverge from each other, Yorke says.

However, at some point, the computer encounters a "glitch" at which successive boxes don't overlap, meaning the true and computed trajectories start to diverge significantly. The errors suddenly refuse to stay neatly boxed. From that point on, no one can certify that the computed orbit remains close to a true one.

Yorke and his colleagues proceed on a case-by-case basis. There is no *a priori* guarantee that their boxing, or error-detection, procedure will work for any given initial conditions. However, for any specific trajectory – for a given function and starting point – the results can be checked for a certain number of iterates using Yorke's method.

For example, Yorke has demonstrated that a true trajectory passes through every one of the millions of iterates producing the array of dots in a figure known as the Ikeda map (see cover illustration). That figure represents the results of iterating an equation describing the electromagnetic field within a ring-shaped laser cavity.

Yorke is also interested in the statistical behavior of the Ikeda map, which shows that trajectories spend more time in some regions (shown as brighter areas) than in others. The trajectory may stay in the bright regions for several thousand iterates, then suddenly escape to the darker halo region for 10 or 20 dots before being pulled back into the light. "One of our mathematical objectives is to try to describe these random escapes," Yorke says.

Although Yorke's work does certify that for 10 million or more points, specific chaotic orbits are real rather than merely numerical artifacts, many questions remain. For example, what's the ultimate behavior of chaotic trajectories when there are infinitely many iterates? For a given function, do different trajectories always form roughly the same pattern of dots?

There are many more questions, Yorke says. Chaos theory is a vigorous and expanding new field. The most important questions may be those that have yet to be asked. □

Escalating Errors

The first few steps in a chaotic process are reasonably predictable. It's only for many steps or in the long term that predictability disappears. The reason is related to the way errors escalate. Chaotic processes are associated with mathematical procedures in which small errors made at successive steps accumulate rapidly to destroy any semblance of a pattern. They are also connected with physical systems in which small uncertainties in initial conditions lead to large deviations in long-term behavior.

Consider the following simple mathematical procedure. Start with a number less than 1, then keep doubling it. Every time the answer is greater than 1, lop off the 1 and retain only the decimal or fractional part of the number.

For example, if the starting number is $1/3$, the sequence goes: $1/3$, $2/3$, $4/3$ (which, according to the rules, would be rewritten as $1/3$), $2/3$, and so on. The sequence has a definite repeating pattern.

Now suppose that the computer or calculator can't handle fractions such as $1/3$. All such numbers must be expressed approximately as decimals rounded off to a certain number of places. Thus, the fraction $1/3$ may be expressed as 0.33. Then the sequence becomes: 0.33, 0.66, 1.32 (rewritten as .32), 0.64, 0.28, 0.56, 0.12, 0.24, 0.48, 0.96, and so on. The sequence no longer has a regular repeating pattern.

Increasing the precision to three decimal places produces the sequence: 0.333, 0.666, 0.332, 0.664, 0.328, 0.656, 0.312, 0.624, 0.248, and so on. By the ninth step, the sequence has again diverged significantly. Increasing the precision to a larger number of decimal places doesn't help very much; the errors accumulate too rapidly.

An error or uncertainty of 3 parts in 1,000 is not uncommon in making measurements or computing the behavior of physical systems. Imagine, for example, a billiard table studded with large cylindrical bumpers in a regular pattern. In this situation, each bounce magnifies any uncertainties in the ball's initial position and speed. The errors propagate so rapidly that anyone attempting to predict the ball's position after the first few bounces, based on the table's geometry and the ball's incoming velocity, would fail. Such a model could apply to the way neutrons scatter from materials or the way atoms bounce around inside a container.

"Small errors in knowledge can grow exponentially with time, making long-term prediction of the future impossible," Yorke says. "Chaos is predictability in the short run but not in the long run."

– *I. Peterson*