A Different Dimension

The fourth dimension's extraordinary mathematical properties perplex mathematicians

By IVARS PETERSON

When mathematicians — normally cautious and meticulous individuals — apply adjectives like "bizarre," "strange," "weird" and "mysterious" to their results, something unusual is happening. Such expressions reflect the recent state of affairs in studies of four-dimensional space, a realm just a short step beyond our own familiar, three-dimensional world.

By combining ideas from theoretical physics with abstract notions from topology (the study of shape), mathematicians are discovering that four-dimensional space has mathematical properties quite unlike those characterizing space in any other dimension. For example, whereas ordinary three-dimensional space, with its familiar notions of height, width and depth, comes in just one variety, four-dimensional space has infinitely many, equivalent basic forms, each with a somewhat different kind of mathematics.

This type of abstract result is practically impossible to visualize. It follows logically from mathematical notions of dimension and space — part of a mental game mathematicians play in their search for patterns and relationships among geometrical structures not just in one, two and three dimensions but in higher dimensions as well.

The startling discoveries about the fourth dimension pose riddles for both mathematicians and physicists. Why does the fourth dimension, and only the fourth dimension, have this strange multiplicity of forms? Are physicists using the right kind of mathematics to study the four-dimensional space-time universe?

"When you compare what's true in four dimensions with what's true in higher or lower dimensions, it's now clear that something very different happens in four dimensions," says Clifford H. Taubes of Harvard University. "The question now is: How bizarre is it really?"

The simplest mathematical spaces are known as Euclidean spaces. An infinitely long line is a one-dimensional Euclidean space. A plane, which has width and depth, is two-dimensional. We think of the space in which we live as three-dimensional.

In general, the term "dimension" signifies an independent parameter, or coordinate. A space has three dimensions if each of its points is completely determined by three independent numbers. For instance, it takes three coordinates, representing longitude, latitude and altitude, to specify the location in three-dimensional space of an airplane above the earth's surface. Similarly, a space has seven dimensions if seven numbers are needed to locate a point in that space.

The coordinates themselves have no intrinsic meaning. Physicists often choose a set of coordinates in which the first three represent independent directions in physical space and the fourth is time, but that is only one of innumerable possibilities. The four coordinates, or dimensions, could just as well be pressure, volume, temperature and mass, or any other set of four parameters. What's important in mathematics is the coordinates themselves, not what they represent.

The term "manifold" covers somewhat more complicated types than Euclidean spaces. Manifolds locally appear "flat," or Euclidean, but on a larger scale may bend and twist into exotic and intricate forms. The Earth's surface resembles a two-dimensional mathematical manifold (or two-manifold). An inhabitant of the Great Plains sees essentially a flat surface, whereas an astronaut orbiting Earth sees the surface of a sphere. Any surface, however curved and complicated so long as it doesn't intersect itself, can be thought of as consisting of small Euclidean patches glued together.

Just as a wildflower guidebook highlights key features such as color and number of petals to help readers distinguish one plant from another, special manifold characteristics, often expressed as numbers or algebraic expressions, help mathematicians tell manifolds apart. Such expressions, known as topological invariants, provide a convenient way of putting manifolds into different categories.

Dimension, the number of coordinates required to specify a point in a given space, is an example of a topological invariant. It's the first level of classification in the world of mathematical manifolds.

Manifolds may also be either bounded or unbounded. A circle is an example of a bounded, or "compact," one-dimensional manifold, whereas a line stretching off indefinitely in both directions is clearly unbounded. The same distinction applies to spaces of any dimension.

In lower dimensions, topologists can imagine a set of "ideal" shapes into which manifolds of a particular dimension can be transformed. For instance, all compact, two-dimensional manifolds resemble a sphere with a certain number of holes. In such a scheme, because a topologist can smoothly transform both a doughnut's surface and a coffee mug's surface into a sphere with a single hole (the archetypal model for this category of two-manifolds), the surface of a doughnut and a coffee mug fall into the same group. That's the basis for the old joke that a topologist is someone who can't tell the difference between a doughnut and a coffee mug.

Botanists can place a particular plant first in a family, then into a genus and a species, making finer distinctions at each step. Similarly, topologists, using appropriate invariants, can also examine in greater detail what manifolds look like and how one may be transformed into another.

Much manifold study concerns the search for more finely tuned invariants that make increasingly subtle distinctions. Because manifolds in higher dimensions are impossible to visualize, these invariants often stand in for the manifolds themselves.

Mathematicians have developed reasonable, workable schemes for classifying manifolds in every dimension except three and four. Dimension three remains a puzzle because mathematicians haven't yet been able to prove that postulated classification schemes cover every conceivable three-manifold. Recent attempts to demystify dimension four reveal it to be a special case with characteristics quite unlike those of any other dimension.

"Dimension four seems to be the hardest because there is just enough room for maximum complications but not quite enough to untangle things," says Robion C. Kirby of the University of California, Berkeley.

The central problem in classifying four-manifolds concerns the technical distinction between topological manifolds and smooth, or differentiable, manifolds. A ball, for instance, has a smooth, continuous surface. A closed, empty box has a continuous surface, but because it has sharp edges and corners, its surface isn't smooth.

The difference is crucial because topologists have ways of mathematically smoothing any sharp edges, creases and jagged features in dimensions one, two and three. For example, the surface of a box can be smoothed out in a reasonable way that transforms it into a sphere.

In other words, there's no difference in these dimensions between topological (the more general category) and smooth manifolds. In five dimensions and higher, manifolds come in both the smooth and crinkly varieties, and mathematicians understand when and how the different types occur. In four dimensions, the distinction between smooth and crinkly manifolds is much more complicated and difficult to sort out.

A similar distinction applies to transformations designed to test whether manifolds belong to the same class or fit into different categories. Like a lump of pizza dough, a topological manifold is very floppy. It's free to be kneaded and distorted.

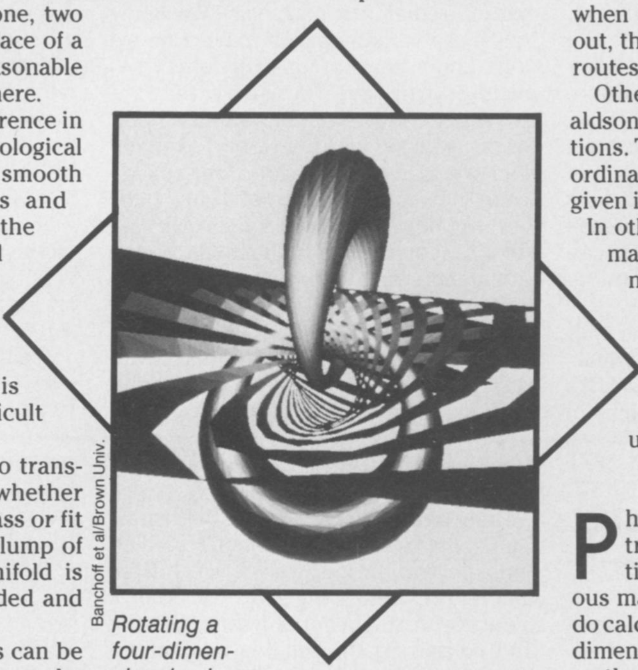
In general, two such manifolds can be regarded as equivalent if one can be transformed into the other without tearing. Such transformations may involve a smooth transition or follow a crinkly course. Calling specifically for a smooth transition is a more stringent condition than simply showing two manifolds are topologically equivalent, somewhat similar to a biologist determining a flower's genus within a family.

The first major step in classifying four-manifolds was a proof that certain types could be identified on the basis of algebraic invariants called quadratic forms. It took Michael H. Freedman of the University of California, San Diego, seven years to crack the problem. His 1981 proof showed that such manifolds can be constructed from simple building blocks and classified entirely on the basis of their quadratic forms.

Freedman's remarkable work unearthed many new examples of four-manifolds and established previously unknown transformations between known manifolds. But it didn't exclude the possibility that some of his four-manifolds may have creases that can't be removed in any way. In other words, dimension four — unlike dimensions one, two and three — may contain manifolds topologically but not smoothly equivalent.

In fact, dimension four turns out to have just such an unexpected feature. In 1982, Simon Donaldson of Oxford University in England proved that not all topological four-manifolds can be constructed in a smooth way. Furthermore, for some manifolds, no amount of tugging or pushing rids the manifold of all its creases.

Donaldson used mathematical tools provided by theoretical physics to prove his point. He worked with a complicated set of mathematical expressions known as the Yang-Mills equations, which had proved critical in physics for predicting the existence of new fundamental particles more massive than the electron.



Banchoff et al/Brown Univ.

Rotating a four-dimensional sphere, which can be visualized as two linked circles and a succession of surrounding doughnut-shaped surfaces, produces a pattern of intertwined rings.

The Yang-Mills equations are notoriously difficult to solve. To find answers, physicists use the known geometric or topological properties of four-dimensional space to get information about potential solutions of the equations.

Donaldson took an even more difficult and daring route by starting with what little was known about solutions to the equations. He used that knowledge to extract information about the underlying four-dimensional space, essentially considering the solutions themselves as mathematical objects to be manipulated according to specified rules he derived.

By showing how to compute quadratic forms from solutions of the Yang-Mills equations, Donaldson demonstrated that quadratic forms are not sufficient to distinguish between topological manifolds that are smooth and those that are not. He subsequently refined his methods to develop new, subtler invariants that distinguish between smooth manifolds even when they have the same quadratic form and are therefore topologically equivalent.

However, the new Donaldson invariants are difficult to define and compute. "We don't understand how they work," says John W. Morgan of Columbia University in New York City. "We're not sure how many we need. Although they work very well in specific cases, we don't have any idea how good they are in distinguishing manifolds in general."

Donaldson also showed that even when four-manifolds can be smoothed out, the process can take many different routes, leading to vastly different forms.

Other mathematicians extended Donaldson's work and took it in new directions. They discovered, for example, that ordinary four-dimensional space can be given innumerable smooth descriptions. In other words, there exist exotic four-manifolds that are topologically but not smoothly equivalent to standard, four-dimensional Euclidean space. In all other dimensions, Euclidean spaces have a unique smooth description, which mathematicians have long used and understand well.

Physicists spend a great deal of time trying to solve differential equations, working with calculus on various manifolds. There is only one way to do calculus in our familiar world of three-dimensional space—a single collection of mathematical expressions that can be treated according to the rules of calculus. When systems of differential equations lead to four-dimensional manifolds, physicists face a puzzling choice. In the realm of exotic four-dimensional spaces, infinitely many ways exist to do calculus. Each exotic space has its own appropriate collection of expressions.

However, the countless exotic differentiation structures in four-dimensional, Euclidean space all involve very special, "cooked-up" behavior, which appears to rule them out as reasonable models for our own physical universe. Physicists deal with spaces that appear feasible. So far, they haven't seen any phenomena that require a step into exotic four-dimensional spaces.

Nevertheless, the existence of these exotic forms does emphasize that there is something different about four-dimensional space. In particular, four-dimensional space is where Einstein's theories must work and where modern physics resides. Like explorers who concentrate on their immediate surroundings and miss the distant mountains, physicists may find their theories, formulated for only a small, well-behaved piece of space-time, may not work on a larger scale.

Another striking feature of these exotic four-dimensional spaces is that they appear to become extremely complicated at great distances and infinitely complex at infinity. In other words, complexity increases as the scale increases. Such a picture may apply to the distribution of matter in the universe.

Indeed, as astronomers study the universe on larger and larger scales, some see hints that the distribution of galaxies and interstellar matter doesn't seem to even out. They detect evidence for irregular arrangements of giant structures, punctuated by large gaps, as far as the aided eye can probe. Astronomer R. Brent Tully of the University of Hawaii in Honolulu, in mapping the distribution of galactic clusters at cosmological distances, proposes that the local supercluster of galaxies in which our Milky Way galaxy resides is actually part of what he calls the Pisces-Cetus complex, a gargantuan grouping of superclusters extending more than a billion light-years.

But Tully's observations lie at the limit of observational work in astronomy, making measurements tough to interpret. It's also difficult to imagine how such an immense, irregularly distributed collection of galaxies could have formed through gravitational effects in the comparatively brief time available since the Big Bang.

Nevertheless, one might expect a universe like the one Tully sees – becoming increasingly convoluted and contorted as more distant regions are explored – if it were embedded in an exotic four-dimensional space rather than a space of the conventional Euclidean variety.

Mathematicians, too, are puzzled by what all this means. Four-dimensional space is indeed strange, with many mysteries yet awaiting solution. For instance, picturing exotic four-dimensional spaces, or four-spaces, remains a problem. "We know these exotic four-spaces exist, but we don't know how to construct them explicitly," Kirby says. "In a sense, they are extremely convoluted. You wouldn't want to do your several-variable calculus homework on such an exotic four-space."

Furthermore, mathematicians have not finished the task of classifying smooth, compact four-manifolds. These researchers have two different pictures of four-manifolds – one in terms of a construction procedure and the other in terms of Donaldson's mysterious invariants. But they haven't yet succeeded in bridging the gap between the two.

"The classification problem is wide open for smooth manifolds," Morgan says. "We now know that it's a very complicated, intricate and delicate classification, but we have no idea even of its general outline."

The methods used for studying four-manifolds also may have far-reaching consequences. Paradoxically, they show that the study of space in five and higher dimensions is simpler and easier to understand than is the study of space in three and four dimensions.

There seems to be a need for new and fundamental insights to aid in understanding four-manifolds. Donaldson's work in particular rests upon deep connections between mathematics and physics. "Nobody knows yet the full power of what Donaldson has done," Taubes says. "We really don't know what's missing."

Perhaps further progress will follow more exchanges between physics and mathematics. Recent work linking quantum field theories and knot theory may be a step in the right direction (SN: 3/18/89, p.174).

"It's a very exciting corner of mathematics," says Ronald J. Stern of the University of Utah in Salt Lake City. "There's a lot going on, and the dust has yet to settle."

"Dimension four is a bizarre dimension," Morgan adds. "But we're beginning to get used to it now." □

News of the week continued from p.327

Spider webs: Luring light may be a trap

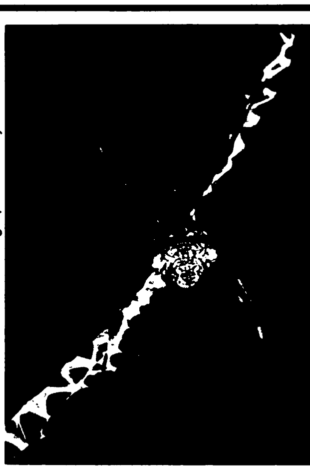
In the first studies of the spectral properties of spider silks, researchers have found that some of these silks reflect ultraviolet (UV) light and that this property lures insects to the webs, says coauthor Catherine L. Craig, evolutionary ecologist at Yale University in New Haven, Conn.

The work "provides an unexpected new insight into the factors that shape the evolution of spider web design," says biologist Stephen Nowicki of Duke University in Durham, N.C. Comparing webs from evolutionarily early and later species, the scientists found that the optical properties of spider webs change with the evolution of the web, Craig says.

In contrast to the traditional view of spiders as passive foragers, the studies also show that "spiders are doing more than we imagined to increase their probability of capturing prey," Nowicki says.Unlike humans, insects can see ultraviolet light and are known to use this sense to locate UV-reflecting flowers and liquids, which may be important food resources or mating sites, Craig explains. But, until now, no one realized that spiders' webs have UV-reflecting properties that turn their prey's UV-detecting ability into a liability – for the insect.

Craig and Gary D. Bernard at the Yale University School of Medicine studied the spectral properties of silks from different spider species by directing a monochromatic beam of light at the silk and measuring the relative amounts of the colors reflected back. They found that the silks from the earliest "ancestral" spiders, which spin silks for only domestic purposes such as lining burrows and covering eggs, selectively reflect ultraviolet light and that the prey-capturing silks of the more recently evolved primitive aerial web weavers, *Uloborus glomosus*, have an even more enhanced UV-reflectance peak.

Robert W. Madden/National Geographic Society



Spider on its web with UV-reflecting designs.

When drosophila fruit flies were given a choice between a *glomosus* web illuminated with white light containing a UV component and one brightened with non-UV-containing light, the majority flew to the ultraviolet-reflecting web. This work indicates that although UV reflectance in spider silk did not evolve for the purpose of capturing prey, its prey-luring advantage seems to have caused natural selection to preserve and enhance the property, Craig told SCIENCE NEWS.

When the researchers looked at the catching silks of the more recently derived garden spider, *Argiope argentata*, they found that the main portions of these webs do not reflect ultraviolet light, but that the decorations added to their webs do. They then discovered that, in nature, decorated webs capture 58 percent more insects than do undecorated webs, suggesting a novel, prey-attracting function for the designs.

Although other scientists have proposed mechanical functions for the designs and recent data from Nowicki and his team suggest that the features function to warn birds of a web's presence, "the strongest point about our hypothesis is that it applies to all situations where you find these [decorative] structures," Craig says. The new studies are scheduled to appear in a forthcoming issue of ECOLOGY. – *I. Wickelgren*