The Color of Geometry

Computer graphics adds a vivid new dimension to geometric investigations

By IVARS PETERSON

Pictures and physical models have long played an important role in mathematics. Nineteenth-century mathematicians, for instance, regularly drew pictures and sculpted plaster or wooden models to help them visualize and understand geometric forms. Their graphic approach represented a way of expressing abstract notions in concrete form. Such sketches and sculptures served as landmarks in the struggle to ferret out the fundamental principles of geometry.

Today's mathematicians are beginning to use computers to create the models they need – converting equations and mathematical structures into colorful, animated images on a video screen. These modern-day pioneers find computer graphics useful for revealing patterns, communicating abstract ideas and suggesting mathematical conjectures worth testing.

One center of such activity is the Geometry Supercomputer Project, based at the University of Minnesota at Minneapolis-St. Paul (SN: 1/2/88, p.12). With access to a Cray supercomputer and the aid of a staff of graphics experts, a select group of 18 mathematicians and computer scientists and their associates in the United States and abroad is breaking new ground in exploring geometric forms and creating breathtaking images of mathematical vistas. Members' interests range from knots and soap-film surfaces to the geometry of hyperbolic space.

Some use computer-generated pictures to study fractals – patterns that repeat themselves on ever smaller scales. Others exploit graphic images to investigate the results of repeatedly evaluating algebraic expressions. Still others look for solutions to geometric problems arising in simulations of a beating heart or a growing crystal.

"Many mathematicians like to sketch things," says Albert Marden of the University of Minnesota, who organized the Geometry Supercomputer Project. The project allows them to go beyond the pencil, he says.

"But it's not as easy as using a pencil," Marden adds. "It's hard to write computer programs, and people often don't have the necessary equipment. In this project, mathematicians for the first time can participate in the world of professional graphics and learn what it has to offer."

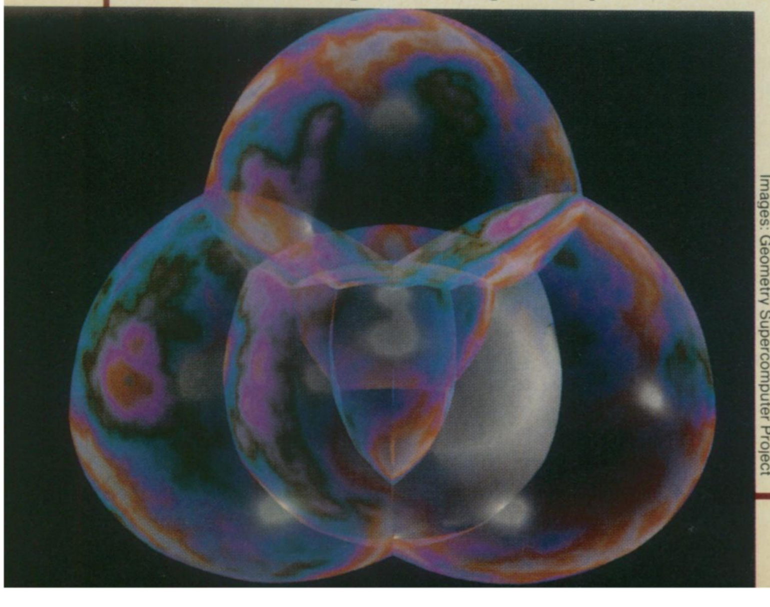
A flight simulator lets you soar over fields and lakes, dodge mountain peaks and explore exotic terrain without ever stepping into an airplane. It all happens at your computer terminal, and you control all the movements.

Now imagine flying into a three-dimensional mathematical structure – a surreal, brightly lit landscape representing

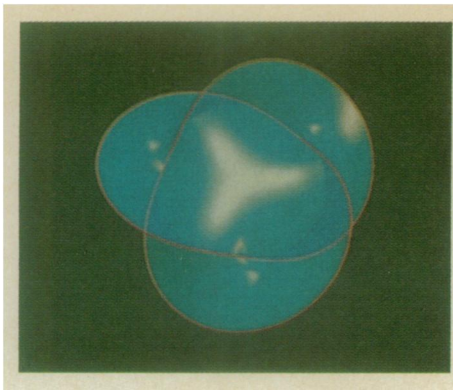
Saddle soap

A close look at the boundary between two adjacent soap bubbles reveals the interface as flat if the bubbles are equal in size, or as a smoothly curved portion of a sphere if one bubble looms larger than the other. But the interface doesn't have to be spherical. Computer experiments reveal that a cluster of six bubbles can include an interface between two bubbles that is saddle-shaped instead.

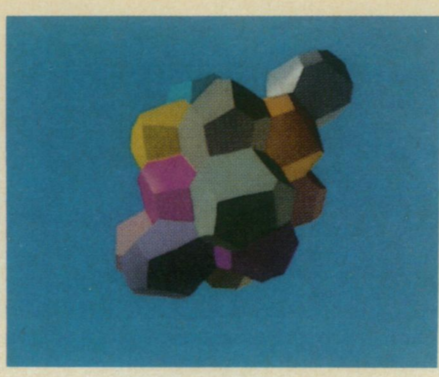
The six-bubble cluster shown, first identified by John M. Sullivan of Princeton University, consists of two small bubbles trapped at the center of a grouping of four larger bubbles. The boundary between the two inner bubbles, each shaped something like a pillow, takes on the unusual saddle form. No one yet has tried to construct such a configuration using real soap bubbles. A six-bubble cluster is the smallest known cluster featuring a saddle-shaped interface. "It's an open question as to whether it can be done with five bubbles," says Princeton's Fred Almgren. "If anybody can do it with five, or suggest how to do it, we'll name the bubble cluster after them and give them a beautiful picture of it."



Images: Geometry Supercomputer Project



The "surface evolver" program can generate a surface that, like a soap film, spans a given twisted wire loop.



The "Voronoi cell evolver" program divides space into a jumble of polyhedrons, then rearranges the boundaries between the shapes to minimize the total surface area of the interfaces.

an abstract world. You're free to examine scenes from different points of view, peek behind objects and zoom in on interesting features — all at your own pace.

That's the idea behind the "hyperbolic viewer" developed by computer scientist David P. Dobkin of Princeton (N.J.) University and his colleagues. This computer program illustrates the striking tools members of the Geometry Supercomputer Project are developing to help mathematicians feel more comfortable using computers to investigate mathematical questions.

Different geometries have different rules. For example, in hyperbolic geometry, the sum of the angles within a triangle is less than 180°, whereas the sum is exactly 180° in ordinary, Euclidean geometry. It's relatively simple to program the hyperbolic viewer to show scenes as they would appear in any of a number of different geometries.

Hyperbolic space in particular provides an unusual but rewarding perspective. "As you fly toward things, you get more and more detail," Dobkin says. For example, a pleated surface patched together from hundreds of triangles opens up to reveal still more triangles. That characteristic of hyperbolic space makes it possible to show lots of detail in one part of a scene without cluttering other parts.

The hyperbolic environment may have value as a medium for displaying abstract structures known as graphs, which are simply sets of points connected by lines. When displayed as spheres linked by tubes, graphs resemble the monkey bars found in playgrounds. Mathematicians can use the hyperbolic viewer to help lay out three-dimensional graphs — initially specified only by sets of relationships between adjacent pieces — in order to seek patterns among the resulting arrangements.

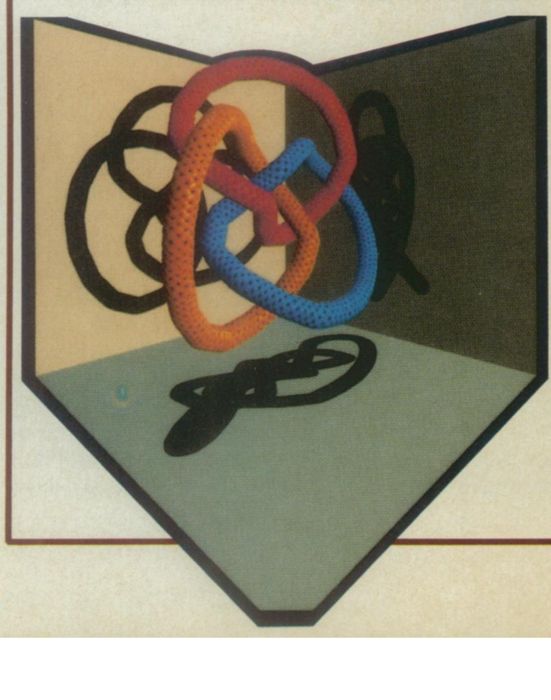
The soap bubble is nature's answer to the problem of packaging a given volume in the least amount of wrapping. Its spherical skin represents the smallest surface area attainable for a fixed volume.

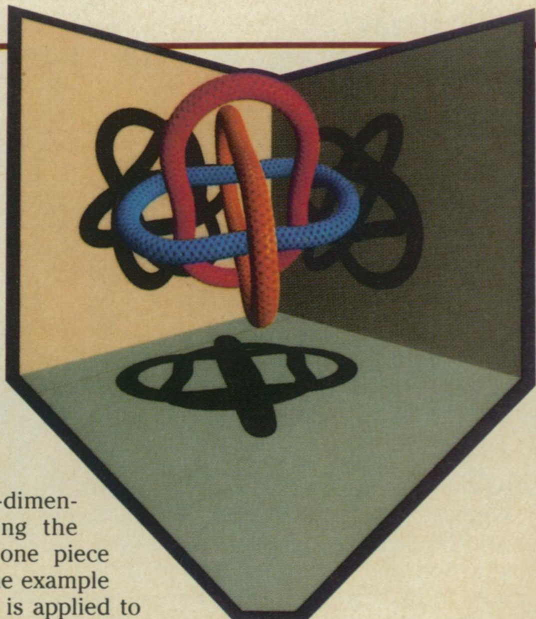
The project's answer to the minimal-surface problem is a computer program known as the "surface evolver." Developed by Kenneth A. Brakke of Susquehanna University in Selinsgrove, Pa., the

Shadows and links

The conventional representation of a mathematical knot involves drawing a continuous loop with breaks in the line to indicate where one piece passes beneath another (SN: 5/21/88, p.328). But such depictions of knots and links (assemblages consisting of two or more intertwined knots) are little more than shadows of the three-dimensional knots themselves, and any knot can appear quite different when drawn in different ways. Knot theorists often have trouble determining whether two seemingly different diagrams actually represent the same knot or link.

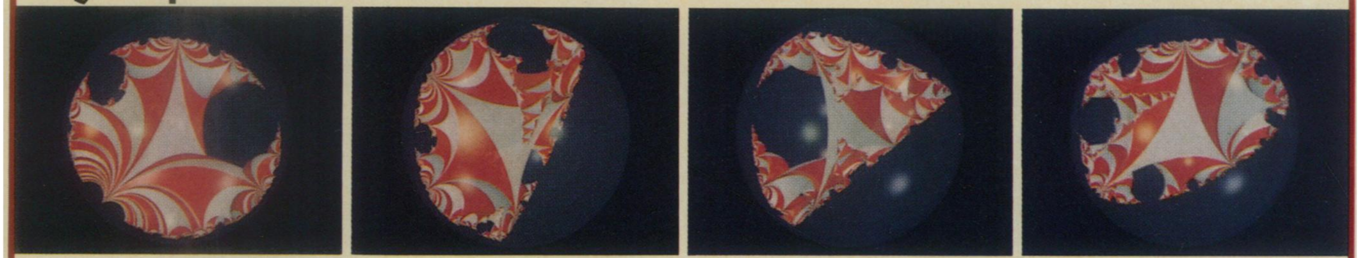
Graphics specialists Charles Gunn and Matt Grayson are experimenting with a scheme that transforms a shadow drawing of a knot or link into a three-dimensional rendering. Then the program proceeds to straighten and smooth out the depicted three-dimensional knot by making the strand behave as if one piece repelled another. In the example shown, the procedure is applied to three linked rings in a configuration known as the Borromean rings. The first picture (left) shows the configuration just after the initial shadow diagram is converted into a three-dimensional form. The second picture (right) shows how the rings have spread out and separated to assume a more symmetric appearance.





Applying the same procedure to two different knot diagrams may help theorists spot whether the two diagrams represent the same knot. "The scheme needs much more work yet," Gunn says. "Still, it reveals interesting symmetries."

Magic carpet



One way of mathematically defining a torus — the surface of an innertube — is to imagine carefully slitting open the torus, then unrolling and stretching its surface to form a rectangle. A slew of such rectangles, placed end to end and side by side, completely carpets the plane.

The same procedure applied to a punctured torus but depicted in hyperbolic space yields a startlingly different pattern. In this sequence of diagrams showing four different views of the mathematical object, a pair of adjacent red and white hyperbolic triangles represents a fundamental region corresponding to the rectangle for an unblemished torus in ordinary, Euclidean space. Mathematicians are exploring several other ways of visualizing such forms.

program is now available to any mathematician interested in generating and studying least-area surfaces.

Starting with a geometric figure enclosing a certain volume, the surface evolver transforms that shape into a new one of the same volume but having the smallest possible surface area. For example, by breaking up its surfaces into successively smaller triangles, the program turns a cube into a close approximation of its least-area counterpart — a sphere. It can also compute the minimal surface spanning any particular wire frame.

Given suitable geometric starting points, the surface evolver computes and displays a wide range of minimal surfaces. Often, there is no equation to express what such surfaces look like, so the computer-generated pictures furnish the best available evidence that a particular form can exist.

"In the past, we simply guessed something would be there or tried it by blowing a soap bubble," says Frederick J. Almgren Jr. of Princeton. "Now it's easy and convenient. You just generate these fantastic geometries on the computer."

The project's minimal-surface team also has a computer program that mimics the behavior of a cluster of bubbles. "You start off with a bunch of points in space, then you divide up the space into cells so that each point has all its near neighbors in its own cell," Brakke says. That creates a jumble of shapes, each having a fixed volume. The program rearranges the boundaries between the shapes, adjusting their geometries without changing their volumes, until the total surface area of all the interfaces reaches a minimum.

The same program now plays a role in efforts to simulate crystal growth, especially the formation of the branching patterns typical of snowflakes. Whether this new approach will work in creating realistic crystal patterns isn't clear yet. "We have no proof yet that the scheme works," says Jean E. Taylor of Rutgers University at New Brunswick, N.J. "But computer experiments will tell us whether we're on the right track."

The heart is a complicated mass of tissue, consisting of bundles of oriented muscle fibers. Some fibers in the heart wall wind around to form shells that look like nested doughnuts. Other fibers take more complicated paths that lace together the heart's right and left halves.

Charles S. Peskin of the Courant Institute of Mathematical Sciences at New York University has spent more than a decade untangling these paths to develop a computer model of a beating heart, first in two dimensions (SN: 9/27/86, p.204) and now in three. "Moving to three dimensions raises interesting and difficult geometric questions," Peskin says. "We

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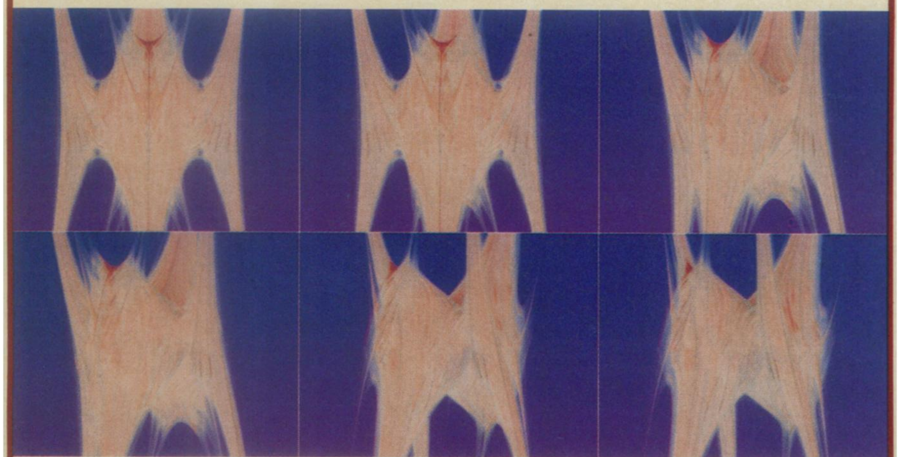
Cubic tissue



Graphic images provide a way of investigating the results of repeatedly evaluating algebraic expressions. The idea is to substitute a certain number into an expression, find the answer, then plug the answer back into the same expression, and so on, to see where the sequence of answers leads (SN: 12/3/88, p.360). For some starting points, the answers get steadily larger, while for others, the answers stay close to the initial values. This iterative process leads to the creation of colorful, intricate portraits of algebraic expressions, many of which show fractal patterns.

Plotting the results of iterating cubic polynomials of the form $z^3 - 3az + b$ for different values of the parameters a and b produces a four-dimensional mathematical object, which marks the boundary between initial values of a and b that when iterated stay near the initial values and those that "escape." This set of diagrams shows different views of the three-dimensional form created by a particular "slice" through this four-dimensional structure. To create a translucent appearance that would enable mathematicians to study both the inside and outside of the resulting form, Charles Gunn applied graphics techniques used in medical imaging.

answer some medical questions.

For one, it hints at a connection between gray hair's ability to carry light and the incidence of a type of skin cancer, he says. The cells in the skin's dermal layer, where hair bulbs lie buried, sometimes become cancerous. Wells observes that the melanin in brown hair and in the scalp can deflect much of the solar ultraviolet radiation that causes this condition, known as basal cell carcinoma. Graying hair represents a reduction or disappearance of pigment, he says. So the question becomes: Does gray hair actually pipe in cancer-causing ultraviolet light to the basal layer, enabling the light to bypass the damage-deflecting mechanisms of the scalp? The question awaits investigation, he says.

That speculation, if confirmed, might be a boon to the hair dye industry. "If you have a cosmetic product that may have a medical side to it as well, in that it cuts down light transmission through hair, then it could become an even better money spinner," Wells predicts, though his letters to several cosmetic companies have so far gone unanswered.

Lakshminarayanan/Enoch

The outer segments of human retinal photoreceptors can guide light in a number of patterns. Shown are commonly observed waveguiding modes.

A student visiting Wells' laboratory helped to uncover another possible example of biological fiber optics, this time in plants. The student wanted to try applying a certain fluorescent staining technique, which Wells originally developed for looking at the cells of human hair bulbs, to the hairs on plant stems. When the two illuminated the dyed plant specimen, which glowed yellow, they found that most of the light emerged from the hairs and not from the nearby cells on the stem, Wells told SCIENCE NEWS.

Though he says he is fascinated by such experiments, the pursuit of natural optical fibers falls outside the scope of his official research program. He doesn't expect to spend much more time on them, especially now that a sizable grant for his biological dosimetry work has been renewed. But he hopes his observations will spark others to investigate the phenomenon.

For Enoch, who has been studying retinal cells for decades, the idea of biological fiber optics is a logical one. "I have surmised that we've had multiple evolutions of waveguides in animal species," he says. "If you look at the hair cells of the cochlea or in the vestibular system [the balance center in the inner ear], they all are cilia modified in different ways. But the ones in the retina turn out to be lightguides in the vertebrate and also have the usual fine tubule structure of a typical hair."

Does this mean that retinal photoreceptors and head hair might be evolutionarily linked in their waveguide function? At first blush, the answer would appear negative, given Wells' speculation that gray hair's fiber-optic properties could threaten an individual's survival. But it's still too early to say whether the light-shunting is merely an incidental effect of pigment depletion, or whether a full head of waveguides might also confer some unknown benefit. That, Enoch says, is a question he'll save for a retirement project. □

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are very close to a three-dimensional, anatomical model of how the fibers are laid out. We're now looking for better ways of visualizing it."

Peskin is particularly interested in the problem of reducing the fiber geometry to sets of numbers a computer can use efficiently for its calculations. Especially challenging is the additional problem of displaying a three-dimensional structure in motion while showing the fluid flow that occurs inside it.

"We can look at our results now, but at the moment the pictures are pretty crude, and we want to improve them," Peskin says. "My hope is that the tools the project is developing will turn out to be useful for this."

When sufficiently refined, Peskin's heart model could allow researchers to study how a normal or diseased heart functions and to use the simulation as a test chamber for experimental devices, such as newly designed artificial valves to regulate blood flow.

While addressing participants' specific mathematical problems, the Geometry Supercomputer Project tackles broader issues as well. "Part of our mission is to develop a graphics programming environment for mathematics," Marden says. Developing computer programs, graphics techniques and tools for other mathematicians to use means establishing compatibility standards so that researchers can readily share software and communicate results. And the search for such standards raises a host of questions concerning how best to represent and manipulate two- and three-dimensional shapes in a computer.

It also means writing computer programs that work on a variety of different machines. "We're trying to save people from having to learn the idiosyncrasies of each new device," Dobkin says.

Adds Charles Gunn, director of the project's graphics laboratory, "Part of the dream is to bring graphics tools to the people who can benefit from them. We're just beginning to see how graphics can be used and how it's going to change the way we do mathematics."

The main uncertainty at present is whether the project, now nearing the end of its second year, will continue beyond its three-year mandate. Currently funded by the National Science Foundation (NSF) with additional support from other sources, the project represents a significant drain on the funds available to mathematicians for research. Some critics grumble that NSF should distribute its scarce resources among a greater number of individual mathematicians rather than providing an expensive, flashy playground for some of the world's top geometers.

Project participants, who regard their effort as a highly successful experiment, hope to extend the project's lifetime by transforming it into an NSF science and technology center devoted to the computation and visualization of geometric structures. Their proposal calls for a budget of nearly $25 million spread over five years.

Group members also envision an important educational role for the tools they're developing. "Computer visualization offers an ideal approach to the teaching of mathematics," they state in their NSF proposal. "Not only the images, but also thinking how to produce the images, are powerful aids to understanding."

Moreover, novel graphics techniques and large-scale computation allow mathematicians to tackle problems they would otherwise find impossible to solve or even consider. "They increase the playing field in which your ideas can operate," Almgren says.

"In some sense, mathematics is the problems you look at as well as the answers you get," Taylor adds. "This approach extends the imagination and opens up many new questions." □