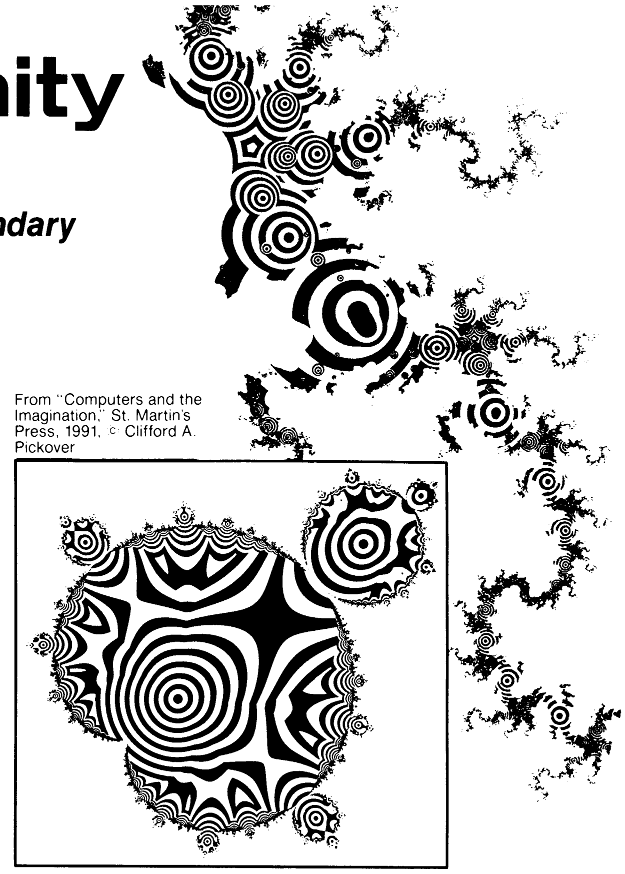
Bordering on Infinity



Focusing on the Mandelbrot set's extraordinary boundary

By IVARS PETERSON

The Mandelbrot set serves as a prime example of how simple mathematical operations can yield astonishingly complex geometric forms. Resembling a hairy snowman, this intricate shape has a convoluted border of loops and curlicues that calls to mind the extravagant ornamentation of baroque designs.

Magnifying the Mandelbrot set's boundary reveals a never-ending array of successively tinier inlets and protrusions. At various positions, the boundary even twists itself into fuzzy, miniature copies of the full Mandelbrot set. In fact, the more you magnify the figure's border, the more complicated it gets.

Often depicted in psychedelic splendor on posters, book covers, T-shirts and calendars, the Mandelbrot set is also the object of serious mathematical research for those exploring dynamical systems. Mathematician Mitsuhiro Shishikura of the Tokyo Institute of Technology has now proved that the Mandelbrot set's boundary is as convoluted as the boundary of a two-dimensional object laid out on a flat surface can ever get. In technical terms, this means the Mandelbrot set's boundary has a fractal, or Hausdorff, dimension of 2. This settles a decade-old question concerning one of the set's chief characteristics.

"It requires a very tricky, detailed analysis to pin this down," says John W. Milnor of the State University of New York at Stony Brook. "You can't establish it by computer experiment."

Shishikura's complicated proof has withstood scrutiny, Milnor adds. "The chances are very good that it's correct."

Anyone equipped with a suitably programmed computer can generate images that approximate the Mandelbrot set. Drawing the figure requires computing the sequence of numbers c, $c^2 + c$, $(c^2 + c)^2 + c, \ldots$ and determining whether the numbers resulting from a particular choice for the initial value c steadily grow larger or instead stay bounded, never rising above a certain value. The Mandelbrot set consists of all choices of c that stay bounded.

Any computer can perform the necessary operations: Square the starting number, and add this product to the starting number to obtain the next number in the sequence. Then repeat the process by squaring the new number and again adding to it the original starting number, and so on. The only proviso is that the type of numbers used for c must be complex numbers. A complex number consists of two parts, which can be pictured as the coordinates of a point plotted on a flat surface, or plane.

From "Computers and the Imagination," St. Martin's Press, 1991, © Clifford A. Pickover

The true complexity of the Mandelbrot set's border region is hard to discern without the use of computer graphics. A computer can create a colored halo around the figure, making the boundary visible, by tracking how rapidly the sequences of numbers resulting from different starting points grow in size. Starting points that lie within the Mandelbrot set produce sequences of numbers that stay bounded, whereas points outside the figure escape to infinity at varying rates, as shown by the use of different colors.

Shishikura proved that the Mandelbrot set's boundary is so convoluted that it appears to have the same dimension as a two-dimensional area even though it is still mathematically a curve — albeit an incredibly wiggly one — and curves ordinarily have no area. He also proved that, excluding certain points on the boundary, the boundary's two-dimensional area itself is zero.

"This means the boundary is as thick as it could possibly be without occupying an area," Milnor says. "He's pinned it down quite precisely."

"This is excellent work," says John H. Hubbard of Cornell University in Ithaca, N.Y. "It's one of many wonderful results that Shishikura has obtained."

The set, discovered in 1980 by Benoit B. Mandelbrot of the IBM Thomas J. Watson Research Center in Yorktown Heights, N.Y., is more than a mathematical plaything. It offers one way of exploring the behavior of dynamical systems, in which equations express how some quantity changes over time or varies from place to place. Such equations arise in calculations of the orbit of a planet, the flow of heat in a liquid, and countless other situations.

"Dynamical systems are the way in which we model the world, usually through differential equations," Hubbard says. "They are of central importance in both mathematics and science."

The Mandelbrot set in some sense encapsulates the behavior of the simplest possible dynamical system: the iteration of a quadratic polynomial. This procedure involves studying what happens when the expression $z^2 + c$ is repeatedly evaluated so that the numerical answer obtained from one step becomes the starting point of the next.

Many mathematicians have already contributed to the detailed investigation of the Mandelbrot set's boundary and its closely related kin, called Julia sets. Indeed, Michael Lyubich of Stony Brook independently obtained proofs, comparable to those of Shishikura, concerning the area of Julia sets.

The key remaining question concerns whether the Mandelbrot set's boundary is "locally connected." If it isn't, then the boundary would have the same characteristics as a figure in the shape of a spoked wheel, in which any small, spotlighted area that doesn't include the hub shows no link between the spokes.

Mathematicians, particularly Jean-Christophe Yoccoz of the Ecole Polytechnique in Palaiseau, France, have taken several important steps toward resolving this question. "There's certainly still a lot to be understood here," Milnor says. "But it now looks likely that the question will be settled before long."

"If [the Mandelbrot set] is locally connected, then essentially we understand it completely," Hubbard says.

And if it isn't, the Mandelbrot set contains within its tortuous boundary an even deeper mystery. □