Knotty Views

Tying together different ways of looking at knots

By IVARS PETERSON

Mathematical research resembles the construction of an intricate framework in a fog-enshrouded environment. Guided by the basic rules of logic, workers use a variety of tools to assemble components of a great but largely obscured structure. Occasionally the fog lifts just enough to reveal unsuspected links between disparate elements.

A decade ago, few mathematicians would have predicted that the study of knots could furnish a unifying thread in mathematical research. But the fog has gradually thinned, revealing a surprisingly extensive web connecting knot theory with various mathematical specialties.

"Knots are turning up all over in mathematics," says Joan S. Birman, a mathematician at Columbia University in New York City.

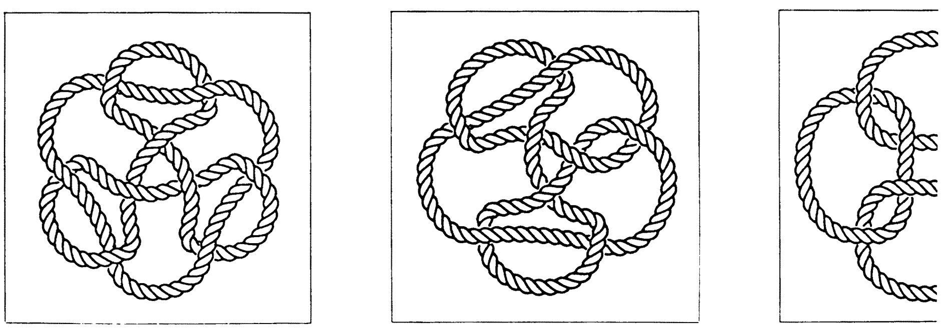
Moreover, new developments in knot theory have provided valuable insights into various aspects of physics, chemistry and biology. In particular, researchers have identified deep connections between the problem of characterizing knots and several areas of mathematics and physics that no one previously suspected had any place for knots (SN: 5/21/88, p.328).

Birman and colleague Xiao-Song Lin of Columbia have added one more strand to this knotted web by bringing new mathematical techniques to bear on the study of knots. Birman described these developments at a joint meeting of the American Mathematical Society and the Mathematical Association of America, held in January in Baltimore.

Compared with the antiquity of many basic ideas in mathematics, knot theory is relatively young. The initial impetus for the systematic study of knots came from a suggestion made more than a century ago concerning the structure of matter. At that time, physicist William Thomson, who later took on the title Lord Kelvin, imagined atoms as minute, doughnut-shaped vortexes of swirling fluid embedded in a pervasive, space-filling medium called the ether.

To explain what distinguishes one chemical element from another, Thomson turned to the notion of a knot. He envisioned atoms of different elements as distinctively knotted vortex tubes. Each twisted tube looked like a knotted rope with its two ends joined together in a loop to keep the knot from coming apart.

Intrigued by this idea, Thomson's colleague Peter G. Tait set out to discover what kinds of knots were possible. This monumental, trial-and-error effort resulted in the first tables of knots, organized according to the minimum number of crossings evident in diagrams of the two-dimensional shadows cast by three-dimensional knotted loops.



David Broman/The Geometry Center, Univ. of Minn.

However, because the same knot can be pictured in two dimensions in many different ways, this undertaking foundered on the difficulty of determining whether the lists were really complete. The researchers had no foolproof method of testing whether two knots, as represented by their diagrams, were the same or actually wound through space in fundamentally different ways.

To solve the problem of distinguishing among knots, mathematicians tried to develop schemes for labeling them in such a way that two knots having the same label are really equivalent — even though their diagrams may appear quite different — and two knots with different labels are truly different.

One such method involves using the crossings in a knot diagram to derive a numerical or algebraic expression that serves as a label for the knot. Such a label, which stays the same no matter how much a given knot may be deformed or twisted, is known as a knot invariant.

In 1984, Vaughan F. R. Jones of the University of California, Berkeley, unexpectedly discovered a connection between knot theory and mathematical techniques that play a role in quantum mechanics. This discovery led to the formulation of a host of new algebraic invariants (or knot polynomials), computed from knot diagrams, that distinguish among knots more effectively than earlier schemes (SN: 10/26/85, p.266), which sometimes gave the same label to knots known on other grounds to be different.

Although mathematicians had recipes for computing these new invariants, they had little sense of what features of three-dimensional knots the resulting algebraic expressions encoded. Even the subsequent discovery of a link between these knot invariants and quantum field theory, which tries to account for interactions between elementary particles, proved unenlightening to many mathematicians (SN: 3/18/89, p.174).

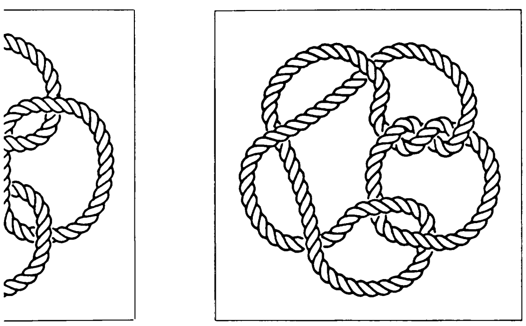
Two years ago, Victor A. Vassiliev of the Russian Academy of Sciences in Moscow introduced a new, radically different way of looking at knots. He started by considering a huge, multidimensional, mathematical "space," in which each point represents a possible three-dimensional knot configuration. If two knots are equivalent to each other, there exists a pathway in this abstract space from one configuration to the other.

This strategy allowed Vassiliev to study not just individual knots but also the ways in which distinct knots fit together. Indeed, his attempt to classify pathways from one knot to its equivalent led to a means of computing numerical knot invariants associated with patterns of connected lines — known as graphs — in which crossed strands in a knot diagram merge to form nodes.

Initially, Vassiliev's approach seemed formidable and impractical. Many mathematicians who read his paper found his techniques very difficult to apply in practice and could see no guarantee that usable knot invariants would emerge from his work.

Birman and Lin, however, discovered a way of translating Vassiliev's scheme into a set of rules and a list of potential starting points. "That's what began to suggest that [Vassiliev's invariants] really looked like the knot invariants we already knew," Birman says.

News of this work brought Dror Bar-Natan, now at Harvard University but then a student at Princeton, into the picture. After several days of discussions with Bar-Natan, Birman and Lin proved that the Jones invariants and several related expressions are directly connected with Vassiliev's knot labels. Bar-Natan discovered simultaneously a remarkable link between his own work on Feynman diagrams — pictures used to



provide an intuitive interpretation of interactions between subatomic particles — and Vassiliev's original equations for computing invariants.

Although this research doesn't completely solve the problem of how to interpret the numerous knot invariants that mathematicians have discovered, it provides a familiar framework within which they can begin to tackle the problem. "It changes an old problem you didn't know how to do into a new, hard problem that's a lot of work," Birman says. "It's a beginning."

"Vassiliev's work provides a very good insight into the nature of knot invariants generally," says Louis H. Kauffman of the University of Illinois at Chicago. "It's entirely possible that all of the invariants we know are built of the building blocks coming out of Vassiliev's picture."

"It gives us another unifying principle for describing knot polynomials," Birman adds. "Instead of one explanation for knot polynomials, we are instead finding multiple explanations and interrelationships, each very beautiful and each opening new doors for investigation." ☐

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toon containing vehicles of three sizes, then asked the computer to simulate the platoon's response to changes in wind and road gradient; emergency stops; and vehicles entering or leaving the pack. With the fuzzy logic controller, the cars seemed to handle these conditions just fine, Frank reported in August 1991 at the Second International Conference on Applications of Advanced Technologies in Transportation Engineering, held in Minneapolis.

Taking a step further toward automated autos, government engineers in Japan and that country's carmakers have built a "Personal Vehicle System" which looks like a Winnebago camper but does its own driving. It uses five TV cameras, with ultrasonic sensors as backups, to see where it is going and to avoid obstacles. Its computers contain maps of the test roads and can plan a route, then instruct the vehicle to drive to any destination on it — turning left or right as necessary — even in the rain or at night.

"I think in 30 years or so, we may see automated driving for freeway use," predicts Akio Hosaka of the Nissan Motor Co. in Kanagawa, Japan.

No doubt engineers will make cars smarter, but they must not make them too smart for the driver. People can process only so much information at one time, says King M. Roberts at the Federal Highway Administration in McLean, Va. They react only so fast to information they get and, for now, drivers must also concentrate on the road ahead. Those factors will limit the usefulness and safety of some navigational aids.

Roberts and his colleagues studied how well 126 young, middle-aged, and older drivers could maneuver along a computer-simulated 26-mile route. The participants used a variety of navigational systems in a range of driving conditions and complications. Three systems displayed maps on dashboard video screens: One showed the driver as an icon moving through a street map; a second added written instructions such as "turn right"; and the third contained arrows that lit up to tell a driver when to turn. Three other systems just talked to the drivers, giving them varying amounts of information. As a check, some drivers navigated with maps in current use today.

The testers could narrow the lanes on the road, create the effect of crosswinds on the vehicle and make the simulated gauges indicate trouble. In addition, they sometimes asked the driver to do simple math en route. During testing, the researchers kept track of speed, reaction time, heart rate, the car's position and, of course, crashes.

Drivers made more errors when they depended on visual aids than when they just listened to instructions. Also, those watching the dashboard screen drove more slowly and missed more warning signals on the dashboard, Roberts and his co-workers concluded in a May 1990 FHA Technical Report.

Establishing a wide margin of safety could result in automated highways packing in too few cars to help smooth traffic flow. For example, long distances between platoons of cars would decrease the chance of platoon pileups, but "such implementation might reduce capacity," says Purdue's Cassidy.

Driverless driving is just one of many aspects of an automated commute. To become reality, other issues need resolving: establishing standards for the vehicles, deciding who should manage and police the new highways, training traffic personnel and finding the money to pay for all the changes.

"The setting of standards is a particularly troublesome problem. There is a desire to have uniformity; on the other hand, you don't want to stifle innovation prematurely," says Thomas B. Deen with the Transportation Research Board.

Finally, liability questions will arise, because accidents will occur no matter how good the technology. "The deployment of IVHS could bring a shift of liability from the driver to the operator or the manufacturer," says Cassidy. Flawed driving may be minimized, but flaws in how the car is built or operated may still lead to crashes.

In addition, the single-commuter automobile represents just one type of transportation. IVHS should encompass buses, trains, and car pools — and that holistic approach requires coordination of other agencies, Clymer says. He envisions smart buses, with voice and data hookups to central traffic controllers and sensors that track passenger load or bus location. Commuters will carry hand-held communication devices that will route them to a bus, car pool, roadway or train, as congestion requires. "Ultimately [car-pool] matching will be done in real time with a moving vehicle traveling along a similar path," he adds.

All that integration will require tremendous improvements not only in data gathering and management, but also in people management. It requires the cooperation of a tremendous number of individuals and agencies. Up to now, however, IVHS has headed in as many different directions as there are interested people and parties. But its promoters want to change that.

"IVHS is well under way in the U.S. and is going to happen, whether we have a plan or whether we do anything collectively about it," says Deen. "But to make it more effective in a longer run, we believe a plan needs to be developed and needs to be used." ☐