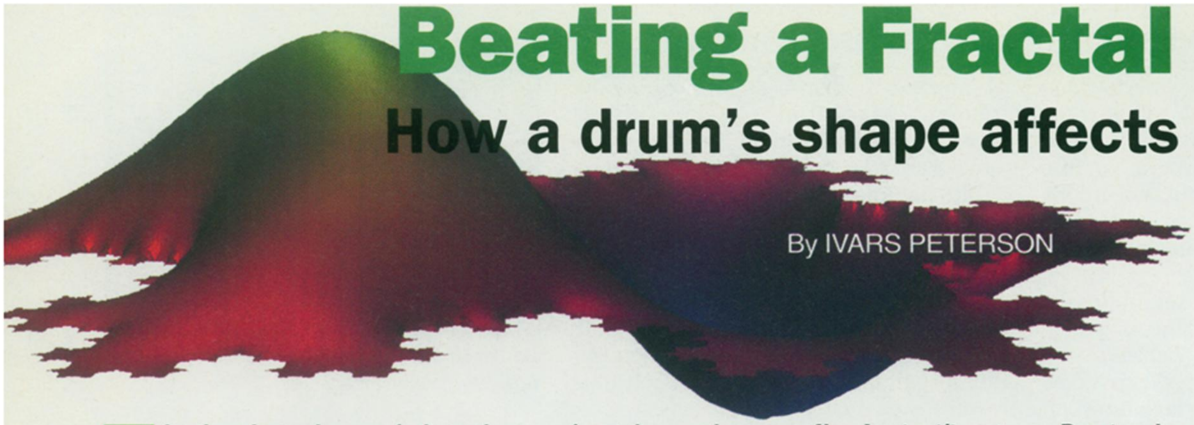


Beating a Fractal Drum

How a drum's shape affects its sound

By IVARS PETERSON

The thundering boom of a bass drum. The rat-tat-tat of a snare drum. The resonant, metallic ring of a kettledrum. It's easy to distinguish the sounds of different types of drums, even without seeing the instruments.

What makes these sounds so readily identifiable is that each drum vibrates at characteristic frequencies, depending mainly on the size, shape, tension, and composition of its sound-generating drumhead. This spectrum of frequencies — the set of "pure tones" produced by a vibrating membrane stretched across a frame — gives a drum's sound its particular voice.

The sounds of drums also suggest important questions in mathematics. Simplified and idealized to its mathematical essence, a drum is a flat, two-dimensional surface held fixed along its rim. Only the interior moves, which greatly restricts the surface's possible motions.

The resulting vibrations, or normal modes, represent the solutions of a mathematical expression known as the wave equation. In the drum's case, the solutions specify the vertical displacement of each point on a surface bounded by a closed curve, such as a circle or rectangle.

Over the centuries, mathematicians have developed ways of solving the wave equation to determine the normal modes of various drum surfaces. They have also pondered the inverse question: Is it possible to infer a drum's shape from a list of its characteristic frequencies?

Recent work on this issue has produced a number of intriguing results. It turns out, for example, that different membrane shapes can sometimes generate identical spectra of frequencies. Thus, in principle, even a person with perfect pitch can't necessarily identify a drum's shape just from its sound.

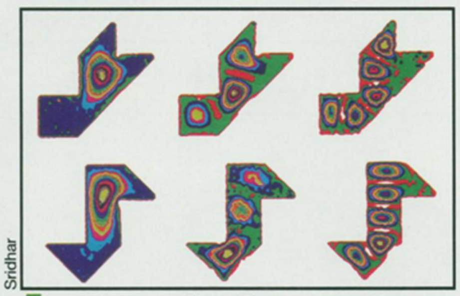
A number of mathematicians and physicists are studying how the wiggliness of a drum's rim affects its sound, especially in cases where the boundary is so wrinkled — with crinkles atop crinkles — that it can be termed a fractal.

"The study of the vibration of drums with fractal boundaries and drums with fractal membranes...[has] significant physical applications to the study of porous media and to that of diffusion or wave propagation on fractals," says mathematician Michel L. Lapidus of the University of California, Riverside.

Lapidus described recent progress in understanding the effects of these curious geometries at a meeting on wavelets and fractals held earlier this year in Pittsburgh.

Such investigations may eventually furnish clues as to why fractals appear to abound in nature, from crazily indented coastlines to the intricate branching of air passages in the human lung.

Moreover, because the wave equation plays a central role in physics, studies of the vibrations of drums under different conditions have important implications for a wide variety of concerns, including the behavior of sound and light, the diffusion of heat, and the workings of quantum mechanics.



Sridhar

These images show the wave patterns observed experimentally in pairs of microwave cavities having different shapes but identical normal modes.

Physicists and mathematicians have long recognized that the shape of the boundary enclosing a membrane plays a crucial role in determining the membrane's spectrum of normal-mode vibrations. In 1966, mathematician Mark Kac, then at Rockefeller University in New York City, focused attention on the opposite question.

Kac asked whether knowledge of a drum's normal-mode vibrations is sufficient for unambiguously inferring its geometric shape. His paper, which proved remarkably influential, bore the playful, provocative title "Can One Hear the Shape of a Drum?"

Previously, mathematicians had established that both the area of a drum's membrane and the length of its rim leave a distinctive imprint on a drum's spectrum of normal modes. In other words, one can "hear" a drum's area and perimeter.

Later, mathematicians also proved that if a membrane has holes, the number of holes can be calculated from the drum's spectrum. But the question of whether one can infer a drum's geometrical shape from its normal modes remained unresolved until 1991.

That was when mathematicians Carolyn S. Gordon and David L. Webb of Washington University in St. Louis and Scott Wolpert of the University of Maryland at College Park came up with two drums that have equal areas and perimeters but different geometrical shapes. They proved that the drums, each a multisided polygon, display identical spectra.

In theory, two drums built out of these different shapes would sound exactly alike. Both would generate the same set of normal-mode frequencies.

Since the initial discovery, Gordon and others have identified many pairs of soundalike drums. All of the known examples have at least eight corners; typically, each member of a pair consists of a set of identical "building blocks" arranged into different patterns.

The existence of these "isospectral" forms indicates that even in relatively simple settings and with a complete set of measurements, it may be possible to reach more than one conclusion. This suggests that similar ambiguities could arise in various physical situations—for example, when geophysicists try to reconstruct Earth's interior from seismic data or when medical researchers generate images of internal organs from X-ray measurements.

"There are things you cannot determine [from a measured spectrum]," says mathematician Dennis DeTurck of the University of Pennsylvania in Philadelphia. "That's the basic message in all this."

It's one thing to prove a mathematical theorem and quite another to demonstrate its reality in a physical situation.

When physicist S. Sridhar of Northeastern University in Boston heard about the Gordon-Webb-Wolpert discovery, he decided to put it to an experimental test — and he had just the right kind of setup to do the necessary experiment.

Sridhar and his coworkers had been investigating aspects of quantum chaos by looking at the patterns created when microwaves bounce around inside thin metal enclosures of various shapes. The same technique could be used to identify normal modes, with microwaves standing in for sound waves and severely squashed cavities standing in for membranes.

To test the drum theorem, the researchers constructed two cavities corresponding to one of the pairs of shapes discovered by Gordon and her colleagues. Fabricated from copper and having eight flat sides, each angular enclosure was nearly 8 centimeters long and less than 6 millimeters thick.

Sending in microwaves through a tiny opening and measuring their strength over a range of frequencies at another location enabled the researchers to establish the frequencies of the normal modes of each cavity.

Remarkably, the frequencies present in both spectra were practically identical. Any discrepancies between the spectra could be attributed to slight imperfections introduced during assembly of the enclosures.

"The day we built this [apparatus] and put the two [spectral] traces on the screen and they lined up so well, I was in awe," Sridhar says. "I had never seen anything like this in many years of building cavities."

"It really was amazing that they could do it," DeTurck comments.

The experiment also provided information that was unavailable mathematically. Although mathematicians had proved that two differently shaped drums can have identical spectra, they couldn't work out the actual frequencies making up such a spectrum. The presence of sharp corners and other special features in the geometries made it too hard to solve the associated wave equation.

"This interplay of mathematics and physics is beneficial to both fields," Sridhar and his colleagues concluded in a report published in the April 4 Physical Review Letters. "While the experiments have provided a satisfying physical basis for the mathematical results, the new ideas from mathematics which have been studied here may have wide and unforeseen impact on physical problems."

A similar link between mathematical results and physical experiments has proved valuable in understanding peculiarities in the behavior of drums with extremely crinkled boundaries and drums having membranes punctured by an infinite array of holes.

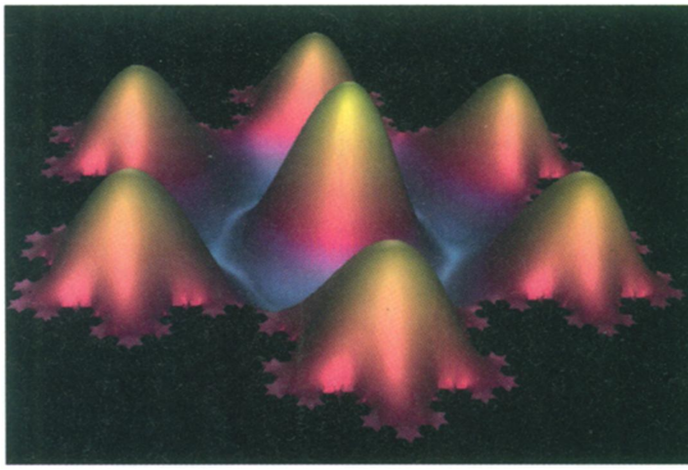
Researchers have long suspected that such fractal boundaries and membranes can drastically alter the types of excitations possible. In 1991, Bernard Sapoval and his coworkers at the École Polytechnique in Palaiseau, France, tested this notion experimentally.

The researchers cut a complicated pattern — resembling a cruciform snowflake — out of a stainless steel sheet, stretching a soap or plastic film across the opening as a membrane. A loudspeaker mounted above the "drum" excited the film, causing it to vibrate.

Sapoval and his coworkers observed that the membrane's convoluted boundary strongly damped the excitations, soaking up their energy. Moreover, waveforms on such a surface showed extreme changes in slope at various locations along the membrane's rim.

But these results apply to a complicated shape that is not really a fractal in the mathematical sense. A true fractal would have a never-ending sequence of tinier and tinier indentations — something not physically realizable.

This computer-generated illustration displays one of the normal modes of vibration of a drum having a fractal boundary approximating the shape of a Koch snowflake.



Griffith, Lapidus, Renka, and Neuberger

Taking a mathematical approach, Lapidus and his colleagues have explored the characteristics of vibrations on membranes with truly fractal boundaries, concentrating on a particular fractal shape known as the Koch snowflake.

The mathematicians have proved several theorems, including one establishing that the slope of a membrane waveform approaches infinity as one gets closer to certain points along the rim. But many conjectures remain unresolved, and new questions keep coming up.

Pictures may provide some clues on how to proceed. Lapidus is now working with graduate student Cheryl A. Griffith and coworkers Robert Renka and John W. Neuberger of the University of North Texas in Denton to display normal-mode vibrations of a drum with a fractal boundary on the computer screen.

The colorful images created so far illustrate vividly the dramatically frilled edges of the waveforms created on fractal-bounded membranes.

However, as in a physical experiment, a computer can only approximate a fractal. It cannot render it in every detail. "We have to be very careful in interpreting the pictures," Lapidus cautions.

One intriguing facet of fractal drums is what they may say about the apparent prevalence of irregular, fractal-like forms in nature. For example, waves can eat away shores to create deep bays and scalloped beaches. At the same time, deeply indented coastlines may survive because they can effectively dampen sea waves, thus suffering less erosion.

"In a nutshell, we observe an irregular ('fractal') coastline because it has a much more stable shape than a smooth one," Lapidus says. This could account for the efficacy of disordered heaps of various-size rocks as breakwaters.

There may also be links between this type of behavior and quantum states in microscopically irregular structures such as porous silicon or in certain types of glass. Sridhar, for example, has looked at the effect of rough surfaces in his microwave cavities.

"We found interesting effects for the spectrum when the surface is rough," Sridhar says. "We saw manifestations of [energy] gaps, just as you would in a solid-state system."

From a mathematical point of view, the relation of geometrical shape to spectrum remains far from completely understood, with more unsolved problems than answers. In many ways, the detective work of deciphering what geometrical information a spectrum holds has barely begun.

"The real problem is to what extent you can reconstruct the medium from the effect," says DeTurck. In the case of drums, what can one tell about these objects from measurements of their characteristic frequencies?

In the orchestra of the mind, it may be possible to identify the distinctively muted, strained tones of a fractal tambourine, but there is no escaping a delicious ambiguity in the beating of different drums. □