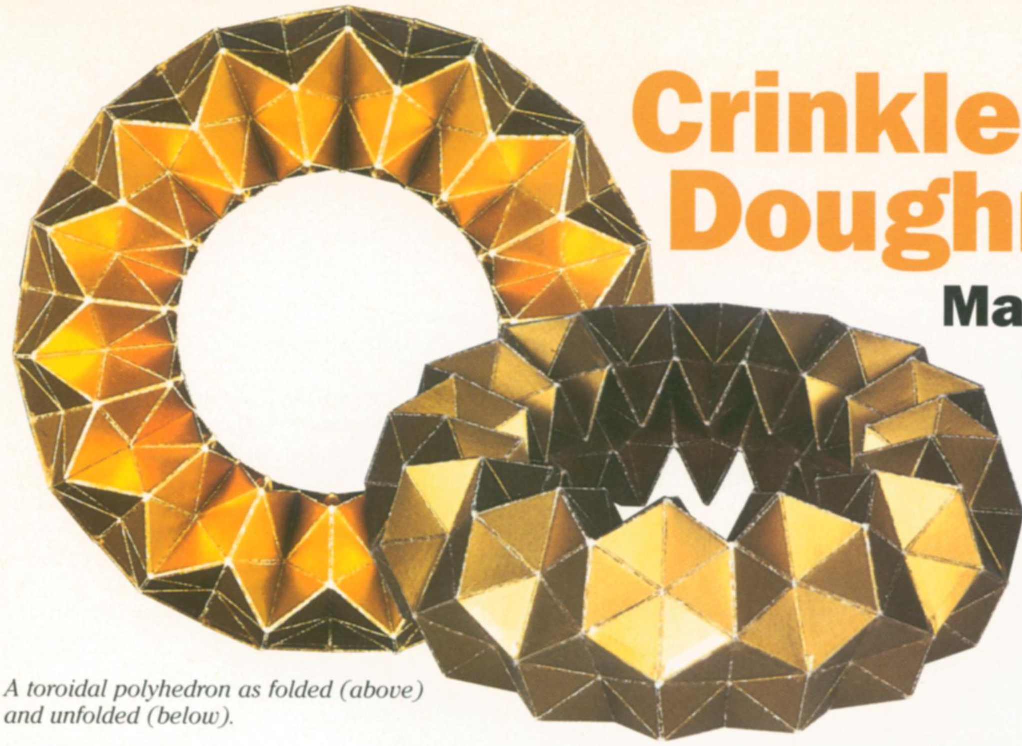
Crinkled Doughnuts

Math in the folds of a polyhedral crown

By
IVARS PETERSON



A toroidal polyhedron as folded (above) and unfolded (below).

Why is a rectangle like an inner tube?

Though this riddle may stump some people, a mathematician specializing in topology can answer immediately.

Imagine the rectangle as a rubbery sheet. Glue together the two longer edges to create an open-ended, cylindrical tube. Then, pulling the tube into a ring, connect the open ends (in effect, gluing together the other two sides). Voila!

But this isn't the only way to turn a rectangle into an inner tube, or torus.

Mathematician William T. Webber has discovered that he can do it without stretching the material. He folds the sheet and tapes its edges together in just the right way to create a polyhedron that looks like a faceted, angular wreath or crown.

As a result, he preserves the flatness of the original rectangle in the individual faces of his toroidal polyhedron. All the surface curvature of an ordinary torus is concentrated in the creases and corners of the polyhedron.

Webber now has a large collection of these intriguingly crinkled rings, made from paper, cardboard, and clear plastic. "As far as I know, I'm the first one to make them," he says.

Their value as a novel kind of origami (SN: 1/21/95, p.44) goes beyond aesthetics, however. The mathematics of these new objects constitutes a significant part of Webber's recent doctoral thesis at the University of Washington in Seattle.

Building things has long been part of Webber's life. At one time or another, he has put together geodesic domes, fiddled with origami, and played with methods of folding sheets of paper to create unusual forms.

"I was aware that the plane could be folded in some interesting ways," Webber says.

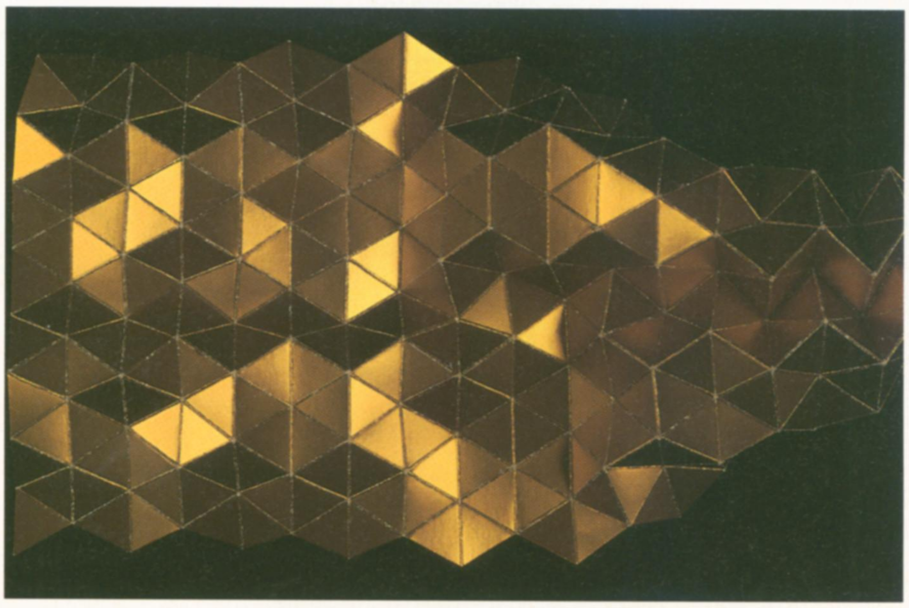
As a student of Branko Grünbaum at Washington, Webber began looking into tilings of the plane. This mathematical pursuit involves the study of patterns created when tiles shaped like triangles, squares, hexagons, or other polygons are fitted snugly together to cover a flat surface without overlapping or leaving gaps.

Webber was intrigued by the possibility of folding along the lines of a section of a tiling—say, a grid of identical triangles—to create some sort of three-dimensional object. In one case, he ended up with a folded shape resembling a hyperboloid—a cylinder with a gently curved waist. "It reminded me of a cooling tower at a nuclear plant," Webber remarks.

He quickly realized that this shape looks a lot like the surface around the central hole of a torus. He asked himself whether it would be possible to close up the folded hyperboloid into a full torus.

Webber's initial evidence that it could be done came in the form of a model constructed from a piece of paper. But that wasn't the same thing as having a mathematical proof establishing the figure's legitimacy.

"Using paper, you can fudge it," Webber says. However, with the help of a computer program to calculate the coordinates of the vertices where triangles meet in his folded rings, he was able to prove that such figures qualify as true toroidal polyhedrons.



Webber's crinkled doughnuts all share certain characteristics: The triangular faces of a given toroidal polyhedron are identical, and six triangles meet at each vertex. Hence, all faces and vertices are equivalent.

This still leaves lots of room for variation, and Webber has expended a great deal of effort trying out different shapes of triangles and different folding patterns to see which ones work.

"I can start with a fold pattern that looks promising and build a model showing that it fits together," Webber says. "Then I can go to the computer and calculate where all the vertices have to be for different triangle shapes."

Over the last few years, he has found infinite families of these crinkled crowns, each with distinctive symmetries and configurations. In some varieties, the top and bottom look different. Others feature helical twists.

More types probably remain to be discovered. "We haven't yet proved that we have found them all," Webber says.

Most of Webber's designs involve triangles with two equal sides. Curiously, he has not yet discovered a pattern of equilateral triangles that leads precisely to a toroidal polyhedron.

"I have one pattern in which the two equal legs of the triangle miss the base's length by less than 0.005 percent," he says. "From a model builder's point of view, that's closer than I can measure. So I use equilateral triangles to make it."

Webber

Webber with different types of folded toroidal polyhedrons he has constructed.

Meanwhile, the mathematical quest continues. "There's a lot of stuff I still don't know about these folded toruses," Webber insists. □

How to fold a crinkled crown

Illustrations: Webber

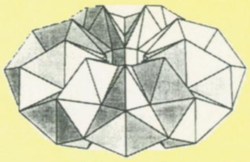


Figure 1

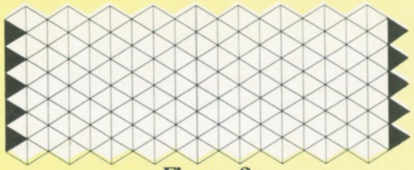


Figure 2

To construct a model of the toroidal polyhedron shown in figure 1, start by transferring the grid in figure 2 to a sheet of fairly stiff paper. Cut it out along its outside edge. The extra triangles along the right side serve as tabs when the edges are glued together in the final step.

Score the paper lightly along each line of the grid pattern, then crease all of the lines in *both* directions.

In the following diagrams, which show only a portion of the full pattern, dot-dash lines indicate mountain folds and dashes indicate valley folds. Thick lines show the folds being introduced at a given step. Unfolded edges appear as thin solid lines.

Fold the paper in a fan pattern, alternating between mountain and valley folds as indicated in figure 3. Fold this up tightly and crease the edges.

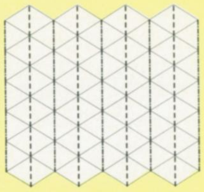


Figure 3

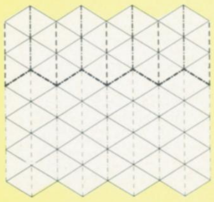


Figure 4

Open the sheet back up and introduce the mountain folds indicated in figure 4. All of the original folds above the new folds must be reversed.

Fold this configuration up tightly to make it look like figure 5, then crease the new folds.

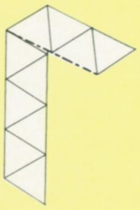


Figure 5

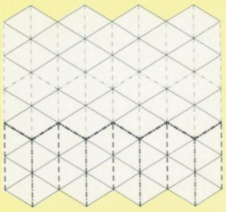


Figure 6

Open up the sheet again and introduce the mountain folds shown in figure 6. This time, all the folds below the new mountain folds should be reversed.

Fold this new configuration up tightly to make it look like figure 7.

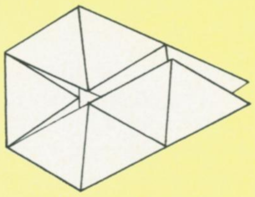


Figure 7

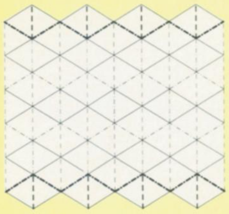


Figure 8

Open the sheet and introduce the final set of mountain folds along the upper and lower edges, as shown in figure 8. Reverse the folds above the new folds on the top and below the new folds on the bottom. Crease the new folds.

With all the essential folding completed, the ends can be glued together, starting with the tabs, to create the basic torus shape. All of the remaining folds should automatically snap into place, and the outside rim should nearly close up. Extra triangles should overlap all the way around the torus; these triangles can be matched with their partners on the opposite side and glued on top of each other.

Similar techniques can be used to construct larger, more intricate models. "With some practice, even some of the very complicated models can be made in only a few hours," Webber notes. — *I. Peterson*