Formulas for Fairness

Applying the math of cake cutting to conflict resolution

By IVARS PETERSON

The marriage had lasted more than 30 years. The wife had given up a promising career as an actress and singer to raise the couple's four sons and manage the household. The husband's business success, in which the wife also played a significant part, had enabled the family to maintain a grand and gracious lifestyle centered in New York City and Paris.

Then, the husband left his wife to live with another woman. In 1981, after a lengthy proceeding, the Supreme Court of New York State granted a divorce. But it took another 2 years of bitter and costly legal wrangling to determine how to divide the couple's joint assets.

According to state guidelines, this case met the criteria for an equal division of marital property, which included a very expensive Paris apartment.

To achieve an equitable split, the court granted the husband all the real estate except for the apartment and ordered him to compensate his former wife for her share of that property. She was obliged to sell the Paris apartment within 3 years of the settlement and divide the proceeds with her former husband.

Stunned by the outcome, she appealed the decision. She prized the Paris apartment and would, at the age of 70, have to seek another home after having lived there for more than 25 years.

She lost her case. However, one of the appeal panel judges did protest that the court-imposed settlement, so meticulously formulated and delicately balanced, was nonetheless unfair to the wife.

Political scientist Steven J. Brams of New York University agrees strongly with the dissenting judge's view. He can also point to potentially fairer methods of handling contentious issues such as the division of marital property.

Brams and mathematician Alan D. Taylor of Union College in Schenectady, N.Y., have worked out mathematical procedures that they claim can be used to settle disputes in ways that both parties see as fair and equitable.

"There are about 1.2 million divorce cases in the United States each year," Brams says. Seldom are both parties satisfied with the provisions of a settlement.

"We have specific procedures that offer very practical solutions in such situations," he asserts.

Nor is divorce the only arena in which these conflict resolution schemes could play a role. Inheritance squabbles, international border disputes, and treaty and contract negotiations could benefit from strategies that promise fair outcomes.

The key to these new methods is the recognition that people generally have different opinions about the values of the items to be shared or the importance of the issues to be settled. These differences make it possible to work out agreements in which all of the parties feel as if they've gotten the best deal. Researchers working on fairness term such allocations "envyfree."

The new methods put together in a practical framework some notions of fairness, equity, and justice that have developed over the last 50 years in philosophy, theoretical economics, and mathematics.

"From the viewpoint of economics, what Brams and Taylor are doing is one case of a more general fair division problem," says economist Hervé Moulin of Duke University in Durham, N.C.

Economists interested in fair division emphasize general principles that underlie the fair allocation of resources, going beyond such precepts as "no envy." They also consider such specific issues as allocating the cost of constructing a shared road or a computer network among potential users, introducing incentives to modify undesirable human behavior, and distributing risk among communities faced with environmental hazards.

"The central issue is how we can get the participants in a scheme of fair division to behave in the right way and not manipulate the system to their own advantage," Moulin says.

Brams and Taylor offer mathematical recipes for solving a particular subset of these problems. "For divorce settlements and things like that, their methods have a lot of potential," Moulin remarks.

When mathematicians ponder fair division, they usually start with a cake.

Suppose a thickly frosted, elaborately decorated birthday cake must be divided among several people. Different people may prefer different parts of the cake—the thickest pink icing for one, strawberry slices for another.

Is there a step-by-step procedure for cutting the cake into pieces so that each participant can guarantee his or her own satisfaction?

There's a familiar strategy for two persons: "I cut, you choose." The first person divides the cake into two pieces that appear equally desirable to him. The pieces may not seem equally desirable to the second person, so she picks the one she prefers. They both automatically end up with a piece that they think is at least

Envyfree Cake Division

Suppose Alice, Bob, and Carol want to divide up a cake. Alice starts by cutting it into three pieces that look equal to her. If Bob views one piece as being largest, he trims it to look equal to the piece he sees as second largest. This leaves one trimmed piece and two untrimmed pieces. Carol chooses one of the three pieces. Bob picks next and must take the trimmed piece if it's available. Alice gets the last piece.

By choosing first, Carol can't lose because she picks the piece she likes best. Bob can't lose because he can choose one of the two pieces he made sure were tied for largest. Alice ends up with one of the two untrimmed pieces, both of which are better in her eyes than the one Bob trimmed.

This procedure can then be repeated with the trimmings until the crumbs are so small that no one cares anymore.

For four players, the cake cutting has to start with an extra piece: Alice must slice the cake into five pieces. The extra piece ensures that no player is forced into taking second best. The number of extra pieces escalates for more people, the researchers discovered. For example, nine pieces are needed for five players, 17 pieces for six, and $2^{n-2} + 1$ for n players.

It's interesting to note that after World War II, Great Britain, France, the United States, and the Soviet Union divided Germany into four zones of occupation, with Berlin, which fell within the Soviet zone, as a valuable "trimming" that was itself divided into four zones. *— I. Peterson* as good as the piece they didn't get.

The Convention of the Law of the Sea, which went into effect in 1994, incorporates such a scheme to protect the interests of developing countries when a highly industrialized nation wants to mine a portion of the seabed underlying international waters. The country seeking to mine would divide that area into two portions. An independent agency representing the developing countries would then choose one of the two tracts, reserving it for future use.

It's somewhat trickier to find an effective divide-and-choose procedure that's fair for three or more people. In the 1940s, mathematicians proved that such allocations are possible but provided no step-by-step method for doing so. By 1960, several such recipes had been developed for the special case of three people, but they didn't work for a larger number.

In 1992, Brams and Taylor invented a cutting and trimming recipe that achieved this sort of cake division for any number of people (see Envyfree Cake Division). Brams and Taylor proved that this envyfree strategy can be carried out in a finite number of steps, at the end of which the cake is completely apportioned, although somewhat mashed. The same routine can be applied in any situation in which the goods are, in principle, divisible into infinitely small amounts.

It isn't always possible to distribute goods by subdividing individual items. Divorce settlements often involve property that can't be split up easily, and some treaty or contract issues have no middle ground.

In recent years, Brams and Taylor have developed procedures for these situations. The starting point is a list of all the disputed items. Working in secret, each combatant ranks these items in order of importance. A mediator can then use the lists to figure out who gets what according to the stated preferences.

To make this allocation, the mediator can follow a procedure that Brams and Taylor call "adjusted winner" (see Divorce Points).

Brams and Taylor have demonstrated mathematically that any allocation arising out of such a scheme is not only equitable but also envyfree. Neither player would be happier with what the other received. Moreover, no other allocation can make one party better off without making the other party worse off.

The adjusted winner strategy is vulnerable to manipulation, however. One party might try to anticipate the other party's rankings and write down scores that deliberately skew the result. In practice, such a course rarely is worthwhile for the schemer unless there's a spy involved who can relay complete and accurate information about the other party's intentions.

"Unless you have the exact information, it's a dangerous game to play," Brams insists. "On the other hand, I don't want to minimize the spite that people have in many of these situations."

Nonetheless, the adjusted winner procedure does a better job in terms of achieving fairness than anything else now available, Brams contends.

Point allocation schemes can also be applied to political disputes and contract negotiations, in which the two sides define the crucial issues, then individually rank their importance.

Brams and NYU colleague Jeffrey M. Togman have recently applied the adjusted winner procedure retrospectively to the 1978 Camp David agreement between Egypt and Israel. In a paper accepted for publication in CONFLICT MANAGEMENT AND PEACE SCIENCE, the researchers used political reports from the 1970s to estimate how Egyptian and Israeli diplomats might have ranked six key issues dividing the two sides. The researchers conclude that their procedure yielded an outcome similar to the agreement actually negotiated.

"This agreement probably could have been achieved more expeditiously, and in a less crisis-driven atmosphere," they argue.

Brams and NYU's David Denoon have tried this scheme on an unresolved dispute between China and several countries of Southeast Asia over claims to the Spratly Islands, a chain of about 230 islets and reefs in the South China Sea. They experimented with different rankings of selected issues, such as what country owns each island, then proposed a methodology—working with two parties at a time—for reaching a fair settlement. The U.S. State Department is now considering use of that scheme in mediating the dispute.

The negotiation literature already contains a great deal of material on identifying crucial issues, articulating them, and splitting them up properly. "Once you've done the hard work of defining the issues, then something like the adjusted winner procedure can kick in to complete the process," Taylor says.

Brams, Taylor, and others continue to explore a variety of mathematical questions that have arisen concerning fair division. For example, they are still looking for a reasonable point allocation system for three or more players.

In cake-cutting schemes, Taylor is intrigued by the fact that strategies involving three people are quite straightforward. But stepping up to four represents a tremendous increase in complexity, with higher numbers only slightly more complicated.

"Mathematically, it indicates there's something there that we don't fully understand in going from three to four," Taylor suggests.

Brams and Taylor recognize that their methods don't offer complete, perfect solutions to human problems.

"The search for better procedures, which make the achievement of fairness not just an outcome but a process as well, will go on," Brams and Taylor write in *Fair Division: From Cake-Cutting to Dispute Resolution* (Cambridge University Press, 1996). "What cannot wait is applying our knowledge, primitive as it is, to problems of fair division that cry out for better and more durable solutions in realistic settings."

This approach represents a modest step toward satisfying a plea made by economist Herbert A. Simon of Carnegie Mellon University in Pittsburgh. "If I were to select a research problem without regard to scientific feasibility," Simon wrote in his 1991 autobiography, "it would be that of finding out how to persuade human beings to design and play games that all can win." □

Divorce Points

In the adjusted winner method of fair division, the husband and wife secretly prepare lists showing how much they value each of the disputed items, ranking them by allocating a total of 100 points among the items.

For example, husband and wife might have come up with these allocations:

Marital property	*Husband*	*Wife*
Paris apartment	33	**55**
Paris studio	**6**	1
New York City coop	**8**	1
Farm	**8**	1
Cash and receivables	5	**6**
Securities	**18**	17
Profit-sharing plan	**15**	15
Life insurance policy	**5**	4
Total	100	100

In this example, based on an actual divorce case, the husband and wife initially win the items that one person rated higher than the other. This gives the husband 45 points to the wife's 61 points. The profit-sharing plan, valued equally by both (15/15), goes to the husband, but that still puts him short of the wife's total. The next closest item is the cash and receivables, which can be divided up so that each party ends up with an equal number of points (in this case, 60.5 points—boldface indicates which party received each asset).

In this way, Brams argues, both husband and wife achieve a more satisfactory result than that actually decreed by the courts. —*I. Peterson*