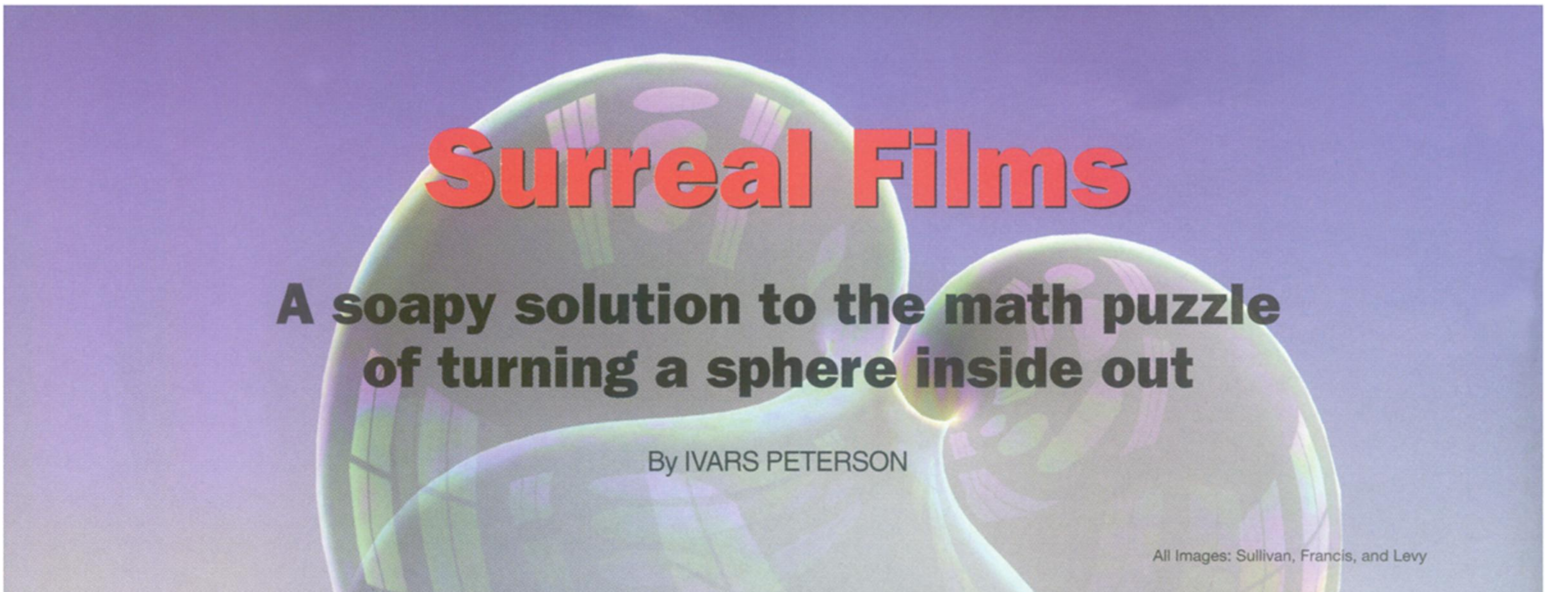


# Surreal Films

## A soapy solution to the math puzzle of turning a sphere inside out

By IVARS PETERSON

All Images: Sullivan, Francis, and Levy

*This eerie bubble represents an early stage in the process of turning a sphere inside out by allowing it to pass through itself without ripping or creasing the surface.*

Whether it's a child climbing a fence or a road crossing a mountain range, the most energy-efficient way to get from one side to the other is to follow the route that passes over the lowest spot.

Mathematicians have now used a similar energy-saving strategy to provide a new, elegant solution to the problem of turning a sphere inside out.

In principle, a determined beachgoer could invert a beach ball by deflating it, pulling the empty bag through its opening, and pumping the ball up again. The task for mathematicians is harder yet: The perfect sphere that they work with has no orifice, and the rules are different.

Imagine a ball made of an extremely thin, delicate, ghostly membrane that can stretch, bend, and pass through itself. The idea is to turn such a sphere inside out without puncturing, ripping, or creasing it.

You could try simply pushing the poles of a sphere toward each other, as if to make them pass through each other and change places. At some point, however, the distorted surface would develop a sharp kink that's not allowed, according to the mathematician's rules.

Avoiding such a kink makes the task of exchanging a sphere's inner and outer surface—called an eversion—a challenging puzzle.

In the past decade, after much effort, mathematicians discovered several ways to accomplish this feat (SN: 5/13/89, p. 299). The latest answer to the problem provides a geometrically optimal path—one that minimizes the energy needed to contort the sphere through its transformation.

"It's not just being able to do it, but being able to do it in the most efficient way possible," says mathematician John M. Sullivan of the University of Illinois at Urbana-Champaign.

The optimal sphere eversion, or optiverse, is now the star of a vivid, computer-generated video, titled *The Optiverse*, produced by Sullivan and his Illinois colleagues George K. Francis and Stuart Levy. It debuted in August at the International Congress of Mathematicians in Berlin.

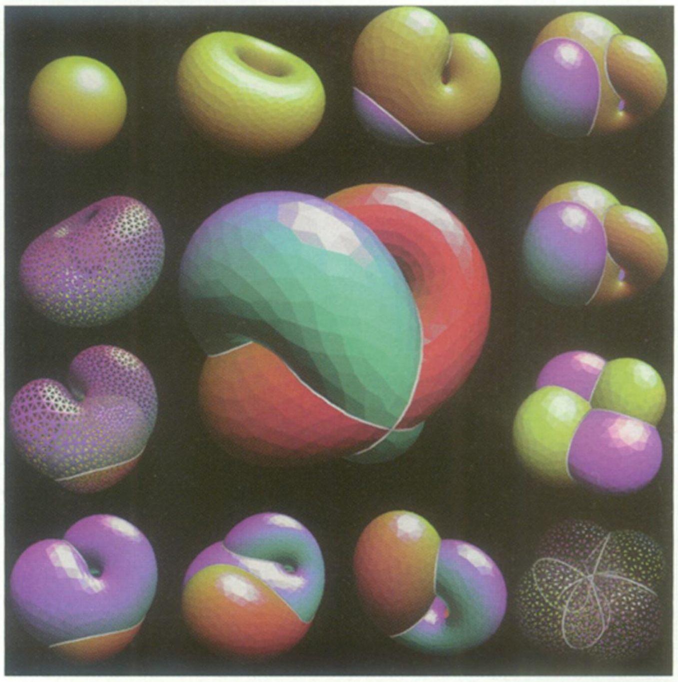
"The sphere eversion problem is perennially intriguing," says Anthony V. Phillips of the State University of New York at Stony Brook. "The optiverse eversion is especially satisfying because it is natural. Given the central, twisted configuration [at the halfway point], the surface unwinds itself."

Until 1957, mathematicians were unsure whether it is possible to turn a sphere inside out without making a hole in or creasing its surface. That uncertainty vanished when Stephen Smale, now at the University of California, Berkeley, proved a theorem in the field of differential topology demonstrating the feasibility of a sphere eversion. The proof furnished no explicit picture of how to do it, however.

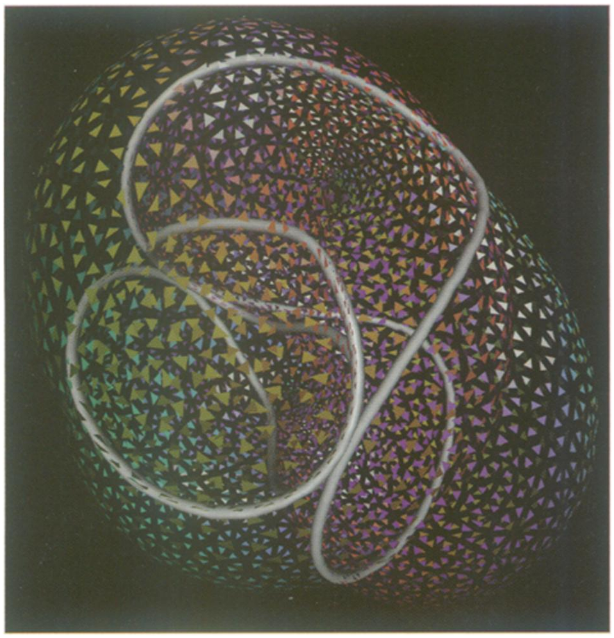
The extraordinary difficulty of visualizing the details of a sphere eversion made it a tremendous challenge to mathematicians and inspired some of the first applications of computer graphics to mathematics.

One early procedure was discovered in 1967 by Bernard Morin, a blind topologist at the Louis-Pasteur University in Strasbourg, France. In 1977, Morin's sphere eversion became the basis of an animated film by Nelson L. Max of the Lawrence Livermore (Calif.) National Laboratory. To create it, Max started with a database of coordinates obtained from wire-mesh models depicting 11 stages in the transformation.

Other mathematicians discovered alternative schemes, gradually simplifying the steps to make the process easier to follow—in effect, telling the story from different geometric points of view (SN:



*In an energy-minimizing eversion (clockwise from top left), a sphere's yellow outer surface trades places with its purple inner surface. A curved white line marks where the surface intersects itself. At several stages, the surface is rendered as a grid of triangles to reveal what occurs inside the contorted sphere. The figure in the center is a close-up of the halfway configuration, showing where four surface sections meet at a point.*



*To monitor key steps during a sphere eversion, mathematicians track changes in the shape of the curve (white) marking where the surface intersects itself.*

6/20/92, p. 404).

William P. Thurston, now at the University of California, Davis, found a particularly striking eversion in which the sphere's initially unwrinkled surface develops a symmetric set of bulges along its equator. Those ruffles enable the sphere to twist around potential kinks as its inside and outside surfaces exchange places. The computer-generated video *Outside In* provided a dramatic view of the entire eversion (SN: 9/2/95, p. 155).

The trouble with these eversions, however, was that they consisted of a series of stages that didn't lead automatically from one to the next.

The new approach was inspired by the idea that a mathematical sphere could be treated like a rubber ball or soap bubble. Adding a little physics to the math, Robert B. Kusner of the University of Massachusetts at Amherst in the early 1980s adopted the notion that such a sphere's surface energy depends on how much bending occurs. A smooth sphere has the smallest overall surface energy, and distortions of the sphere increase that energy.

During any sphere eversion, the surface energy rises to a maximum, then falls back to its minimum when it achieves the inside-out spherical shape. Moreover, topologists have already proved that every eversion must pass through a distorted shape in which at least four sections of the surface meet at one point.

Kusner discovered a surface that resembles the halfway point in the original Morin sphere eversion, when the inside and outside are equally exposed. This shape also has the smallest surface energy of any formation in which four surfaces meet at a point. He conjectured that this complicated, self-intersecting surface might correspond in terms of energy to a saddle point—like a pass through a mountain range. Going downhill from that midpoint configuration to surfaces of lower energy, one could travel in one direction to end up with the original sphere or in another direction to end up with the inside-out sphere.

In effect, all one would have to do is nudge the intermediate surface, and it would automatically snap into its original or inside-out configuration.

Deciding whether a sphere eversion based on this principle would work, however, required additional research. "There were three possible worries," Sullivan notes.

It was possible that the energy of Kusner's surface put it at the bottom of a hollow in the middle of a mountain pass. In that case, a big push would be needed to get that configuration out of its local depression and on its way to the original or the everted sphere.

Another possibility was that all routes would lead to the original, but not the inside-out sphere, or vice versa. Finally, the downhill, energy-minimizing route might pass through an illegal, pinched configuration .

When Kusner conceived of his sphere eversion, there was no graphics software for modeling the type of energy-driven transformations that he had in mind.

More than a decade later, the availability of powerful computers and interactive software for studying surface shapes finally made it feasible to test his approach.

The Surface Evolver computer program, created by Kenneth A. Brakke of Susquehanna University in Selinsgrove, Pa., uses energy-minimizing principles to enable researchers to find the contours of soap films, explore the geometry of foams (SN: 3/5/94, p. 149), and study how shapes change under a variety of circumstances (SN: 12/23&30/89, p. 408). Brakke and Sullivan added constraints to the Evolver's procedures, enabling the software to simulate a mathematical sphere as it everted.

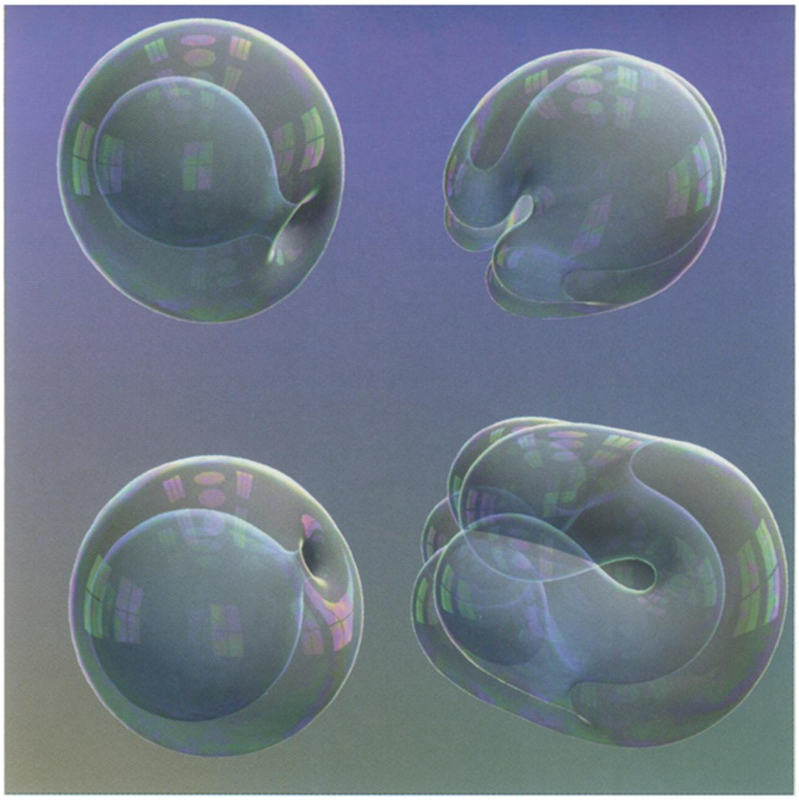
"We did not know ahead of time that the Evolver would be successful in producing a sphere eversion," Sullivan says.

The program represents surfaces as grids of triangles, and the software computes how each of the grid's vertices moves to minimize surface energy. Each frame of the resulting video uses from 1,000 to 2,000 triangles to approximate the contorting sphere's curved surface.

Sullivan and his coworkers visualized the eversion in a variety of ways, cutting away parts of the surface or making the sphere transparent to reveal its inner contortions. The visual effect was unearthly: a weirdly shaped bubble performing a surreal gymnastic routine.

Mathematically, the computations and resulting animation provided evidence that Kusner's eversion works. The midway surface configuration appears to represent a true saddle point.

Additional computer experiments revealed that the case in which four surface sections meeting at one point is just the simplest member of an infinite family of eversions, each one characterized by a different number of intersecting surface sections at the crucial halfway mark of the eversion.



*Several steps of a sphere eversion in which eight surface sections meet at a single point during the midway stage (lower right).*

The energy-minimizing approach pioneered for the sphere eversion may prove useful for other transformations, such as turning the surface of a doughnut (or torus) inside out. Maybe, Sullivan suggests, there's an efficient way to perform that conversion.

However, many of the mathematical theorems describing the surface energy of a sphere have not yet been fully established for a torus. "I'm not sure where to start looking for the unstable critical tori from which I could push off to go downhill," Sullivan notes.

Sullivan can show, for example, that for some starting surfaces, the downhill path leads to pinching. "Perhaps it is just luck that the sphere eversion avoids this pitfall," he remarks.

So, the quest for shape shifters isn't over. It's likely that the sphere-eversion problem and its variants will continue to serve as a rich testing ground for mathematical ideas and computer graphics techniques. □