Fibonacci at Random

Uncovering a new mathematical constant

By IVARS PETERSON

I t all started out with imaginary rabbits. In a book completed in the year 1202, mathematician Leonardo of Pisa (also known as Fibonacci) posed the following problem: How many pairs of rabbits will be produced in a year, beginning with a single pair, if every month each pair bears a new pair that becomes productive from the second month on?

The total number of pairs, month by month, forms the sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, and so on. Each new term is the sum of the previous two terms. This set of numbers is now called the Fibonacci sequence.

Fibonacci numbers come up surprisingly often in nature, from the number of petals in various flowers to the number of scales along a spiral row in a pine cone. They also arise in computer science, especially in sorting or organizing data.

Amazingly, the ratios of successive terms of the Fibonacci sequence get closer and closer to a specific number, often called the golden ratio. It can be calculated as $(1 + \sqrt{5})/2$, or 1.6180339887 . . . . For instance, the ratio 55/34 is 1.617647 . . . , and the next ratio, 89/55, is 1.6181818 . . . .

Now, computer scientist Divakar Viswanath of the Mathematical Sciences Research Institute in Berkeley, Calif., has taken a fresh look at Fibonacci numbers and unexpectedly discovered a new mathematical constant: the number 1.13198824 . . . . He describes his result in a paper to be published in MATHEMATICS OF COMPUTATION.

Viswanath's research represents an intriguing gateway to heavy-duty mathematics, says mathematician Keith Devlin of Saint Mary's College of California in Moraga. It relies on powerful mathematical techniques that also are used, for instance, to elucidate the behavior of disordered materials.

V iswanath wondered what would happen to the Fibonacci sequence if he introduced an element of randomness.

Here's one way to proceed: Start with the numbers 1 and 1, as in the original Fibonacci sequence. To get the next term, flip a coin to decide whether to add the last two terms or subtract the last term from the previous term.
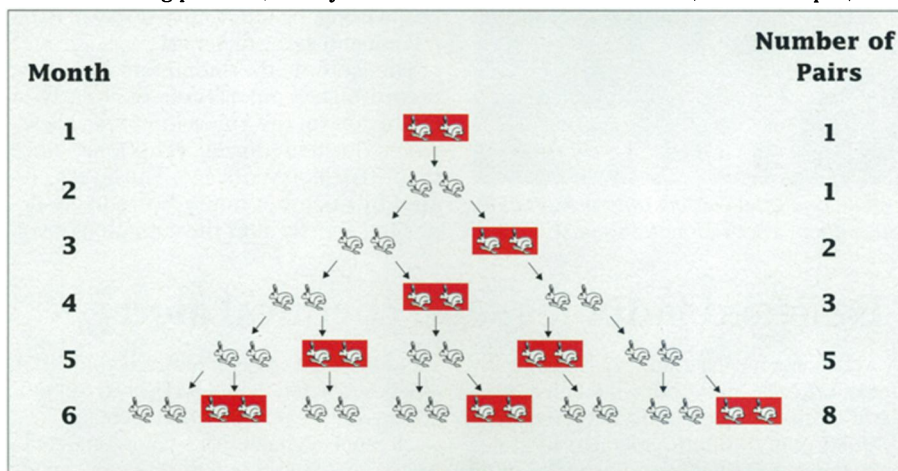
Suppose that heads means add and tails means subtract. Tossing heads would result in adding 1 to 1 to get 2, and tossing tails would lead to subtracting 1 from 1 to get 0. According to this scheme, the successive coin tosses H H T T T H, for example, would generate the sequence 1, 1, 2, 3, –1, 4, –5, –1.

It's easy to write a short computer program to generate these random Fibonacci sequences, notes Lloyd N. Trefethen of the University of Oxford in England. Looking for patterns and trends among such sequences of numbers can be a fascinating pastime, he says.

Fibonacci's immortal-rabbit problem. A red rectangle designates a newborn pair, which doesn't produce offspring until the second month.

Indeed, infinitely many sequences follow Viswanath's rule. A few have special characteristics. If the coin always comes up heads, for instance, the result is the original Fibonacci sequence. Other strings of coin tosses can produce a repeating pattern, such as 1, 1, 0, 1, 1, 0, 1, 1, 0, and so on.

Nonetheless, such special cases are sufficiently rare among all possible sequences that mathematicians ignore them.

The standard Fibonacci sequence has an intriguing property. The hundredth Fibonacci number, for example, is roughly equal to the hundredth power of the golden ratio.

By examining typical random Fibonacci sequences based on coin tosses, Viswanath uncovered a similar pattern. He ignored the minus signs, thereby taking the absolute value of the terms. He found that the hundredth term in such a sequence, for example, is approximately equal to the hundredth power of the

number 1.13198824 . . . . In fact, the higher the term, the closer its value gets to the appropriate power of 1.13198824 . . .

Despite the element of chance and the resulting large fluctuations in value that characterize a random Fibonacci sequence, the absolute values of the numbers, on average, increase at a well-defined exponential rate.

It's not obvious that this should happen, Viswanath observes. Random Fibonacci sequences might have leveled off to a constant absolute value because of the subtractions, for example, but they actually escalate exponentially.

Providing a rigorous proof of the result was a tricky business. To get the answer he required, Viswanath had to delve into several different areas of mathematics, including the intricacies of geometric forms known as fractals, and finish with a computer calculation.

Viswanath's achievement "showed persistence and imagination of a very high order," Trefethen remarks.

Now, Devlin adds, "mathematics has a new constant." No one has yet identified any link between this particular number and other known constants, such as the golden ratio.

S urprisingly, Viswanath's constant provides one answer to a mathematical puzzle that arose several decades ago from the work of Hillel Furstenberg, now at Hebrew University in Jerusalem, and Harry Kesten of Cor-

nell University.

In a different mathematical context involving so-called random matrix multiplication, Furstenberg and Kesten had proved that in number sequences generated by certain types of processes having an element of randomness, the value of the nth term of the sequence gets closer to the nth power of some fixed number. However, they provided no hint of what that fixed number might be for any particular sequence.

Because random Fibonacci sequences fit into this category of sequences, Vis-wanath's new constant represents an accessible example of these fixed numbers.

"It is a beautiful result with a variety of interesting aspects," Trefethen says. It's a nice illustration, for example, of how a random process can lead to a deterministic result when the numbers involved get very large.

Moreover, although Viswanath's result by itself has no obvious applications, it serves as an introduction to the sophisticated type of mathematics developed by

Furstenberg, Kesten, and others. That mathematical machinery has proved valuable in accounting for properties of
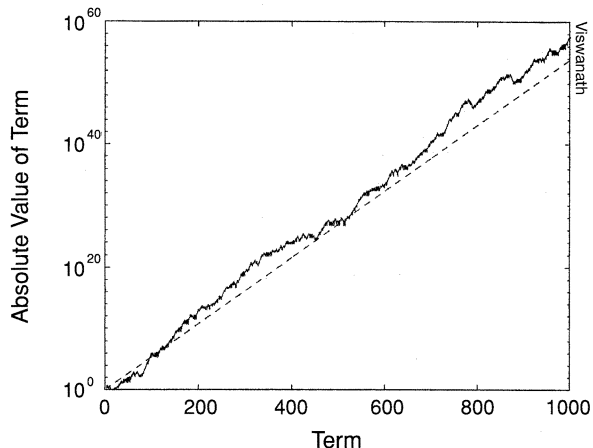
Despite significant fluctuations, the absolute values of the first 1,000 terms of a typical, computer-generated random Fibonacci sequence show a clear trend to larger values, fitting a pattern of exponential growth (dashed line).

disordered materials, particularly how repeated random movements can lead to orderly behavior, Devlin says.

Such mathematics underlies explanations of why glass is transparent

and how an electric current can still pass in an orderly fashion through a semiconductor laced with randomly positioned impurities.

Viswanath's original work was done at Cornell University, under Trefethen's supervision. Trefethen and Oxford's Mark Embree have recently studied slightly modified versions of the random Fibonacci sequence. If, for example, one combines the last known term with half the previous term, adding or subtracting according to the toss of a coin, the sequence's numbers decrease at a certain rate, displaying exponential decay.

By using fractions other than one-half, it's possible to find fractions for which one gets exponential decay, exponential growth, or merely equilibrium. "All this quickly gets under your skin when you start trying it on the computer," Trefethen says, adding that it becomes an addictive pastime.

There's still plenty of room for mathematical exploration and experimentation in a problem that began centuries ago as a decidedly unrealistic model of a population of rabbits. □

Biology
From Chicago at the annual meeting of the American Society for Microbiology

A toxin at the heart of Lyme disease?

Although scientists recently sequenced all the genes of *Borrelia burgdorferi*, they're still struggling to explain how this spiral-shaped bacterium causes the varied symptoms of Lyme disease. Beginning as a simple rash, Lyme disease can go on to cause fatigue, paralysis of the face and limbs, headaches, and even behavioral changes. Sam T. Donta of Boston University School of Medicine and his colleagues now report that the microbe may produce a nerve cell toxin similar to ones made by the bacteria that cause botulism and cholera.

The initial computer-aided scan of the *B. burgdorferi* genome did not reveal obvious nerve-cell toxins, but such searches are notoriously unreliable, notes Donta's colleague Mark J. Cartwright. Consequently, the researchers used strands of the DNA of known bacterial toxins to fish out similar genetic sequences within the Lyme bacterium. They found a gene for a possible toxin and showed that its protein induces cells to become rounder, apparently by altering their internal skeleton.

Eventually, nearly 40 percent of cells exposed to this protein die, reports Cartwright. The next task is to prove that the bacterium actually secretes the toxin during an infection, he says.   —J.T.

One small bacterial genome, to go

Before attempting to create a new life-form in the laboratory, one group of scientists is pausing to allow reflection on the consequences of such an effort. Their experiment concerns the intriguing issue of how few genes a free-living organism needs (SN: 9/28/96, p. 198).

The bacterium with the smallest known genome, *Mycoplasma genitalium*, has just under 500 genes. Scientists at The Institute for Genomic Research (TIGR) in Rockville, Md., have been deactivating those genes one by one to determine which are required for life. The final tally stands at around 300 essential

genes, says J. Craig Venter, former president of TIGR.

To test whether these 300 or so genes alone can sustain life, Venter and his colleagues have contemplated making an artificial bacterial chromosome containing only those genes. The biologists would then slip this chromosome into an empty bacterial membrane and see whether their creation springs into action. The researchers have temporarily halted their work to give a group of bioethicists and religious scholars time to develop a report on the implications of the provocative experiment.

There's no delay, however, in Venter's controversial plan to quickly sequence all the genes in the human body (SN: 5/23/98, p. 334). His new company, Celera Genomics in Rockville, Md., could have the complete human genome next summer, he says.   —J.T.

DNA may reveal a fly's favorite eatery

In the tropics of the Americas and elsewhere, beware the bite of sand flies. These insects, which feast at night, often transmit disease-causing parasites when they procure a blood meal from their victim. Sand flies can infect people with the parasites, called leishmaniase, that cause a family of potentially fatal diseases that afflict an estimated 12 million people worldwide.

The source of the parasites must be another mammal that the sand fly bites. Identifying those natural hosts might help researchers devise better ways to control the spread of the disease. Cristian Orrego of San Francisco State University and his colleagues are using genetic evidence to tackle that challenge. The scientists have already shown that they can isolate hamster DNA from the flies' blood meals after they allow the insects to feed upon the rodents in the laboratory. They now plan to capture sand flies in Peru, isolate mammalian DNA from the insects, and compare it with the DNA of mammals known to reside in the tropics.   —J.T.