SN EDUCATOR GUIDE

April 14, 2018

Stephen Hawking's Legacy Will Live On

Activity Guide for Teachers: Escaping from a Black Hole

Purpose: To better understand black holes and Hawking radiation by deriving expressions and calculating theoretical data that relate to these phenomena.

Procedural overview: Using the basic principles of general relativity and quantum mechanics, derive an expression, then calculate the Schwarzschild radius of a black hole and the temperature of the black hole's Hawking radiation. As an advanced extension, using calculus, derive the expression, then calculate the approximate amount of time required for a black hole emitting Hawking radiation to evaporate.

Approximate class time: One class period.

Supplies:

- Student handout: Escaping from a Black Hole
- Scientific calculators

Directions for teachers: Students can work through this activity in class, where you can offer help if they do not understand something. If class time is limited and the students are sufficiently advanced, they could do this activity as homework instead. Students could work individually, although it may be helpful for them to work in groups to discuss their reasoning along the way. Suggested student responses for correctly derived equations and calculated answers are given in italics below.

Directions for students: In Part A, you can derive an expression for the size of a black hole (the Schwarzschild radius of its event horizon, the classical point of no return for everything including light) with a given mass. Note that even though you will use a classical physics derivation and not special or general relativity, your answer is still an excellent approximation.

In Part B, you can derive an expression, then calculate the temperature of Hawking radiation emitted by a black hole. Matter or energy may escape from black holes in the form of Hawking radiation. According to quantum mechanics, pairs of virtual particles and antiparticles continually appear throughout space and then promptly annihilate each other and disappear. Hawking radiation occurs when such a particle-antiparticle pair appear just outside of a black hole's event horizon. Stephen Hawking theorized that when one member of the pair crosses the event horizon, becoming trapped in the black hole, the remaining particle (or antiparticle) is emitted by the black hole.

A hot object emits photons with a spectrum of frequencies governed by how hot the object is. As an object heats up, it emits infrared photons, followed by red light, yellow light, blue light, ultraviolet light and so forth. This is called black body radiation because the color of the radiation is due to the temperature of the object, not the object's inherent color. The emission of Hawking radiation makes a black hole appear to be radiating like a black body at a certain temperature.

In Part C, you can calculate the time it would take for a black hole to evaporate. Theoretically, black holes that emit Hawking radiation also lose mass and can eventually disappear.

Part A. Calculating the Schwarzschild radius of a black hole

1. Newton's gravitational constant is $G \approx 6.67\times10^{-11}$ meters^{3}kilograms^{-1}second^{-2}. If a small object of mass m (in kg) is separated from a large object of mass M (in kg) by a distance of r (in m), what is the gravitational potential energy (in Joules) of the small object? This potential energy is relative to being infinitely far away from the large object, $r=\infty$, with zero gravitational potential energy.

$$P.E. = -G M m/r$$

2. If the small object is moving away from the large object with a velocity v (in m/sec), what is the kinetic energy of the small object (in J)?

$$K.E. = m v^2/2$$

3. What is the total energy of the small object when it is a distance r from the large object and has a velocity v?

$$E = K.E. + P.E. = (m v^2/2) - (G M m/r)$$

4. If the small object coasts away from the large object until it slows to a stop, and it just barely escapes from the large object's gravity ($r=\infty$) without falling back, what is the total energy of the small object?

$$E = K.E. + P.E. = 0$$

5. By equating the small object's total energy before and after escaping from the large object, what escape velocity v_{escape} must the small object have when it is a distance r from the large object in order to eventually just barely escape?

$$v_{escape} = (2 G M/r)^{1/2}$$

6. A black hole has such strong gravity that even light (traveling at velocity $c \approx 3.00\times10^8$ m/sec) cannot escape if it gets too close. The radius of a black hole's event horizon, the point at which light can no longer escape, is called the Schwarzschild radius R_S.. Everything within that radius appears black to an outside observer. If the escape velocity becomes the speed of light, $v_{escape} = c$, at the event horizon, $r=R_S$, what is the Schwarzschild radius?

$$R_S = 2 G M/c^2$$

Our sun's mass is $M_S \approx 1.99\times10^{30}$ kg, so the black hole's mass M can be expressed in multiples of solar masses:

$$M = (M/M_S)\ 1.99\times10^{30}\ \text{kg}$$

7. Plugging in that expression for mass as well as the numbers for G and c, how would you express the Schwarzschild radius in multiples of solar masses?

$$R_S \approx (2)(6.67\times10^{-11})(1.99\times10^{30})/(9.00\times10^{16}) \quad (M/M_S)\ m$$

$$\approx 2950\ (M/M_S)\ m \quad = \quad 2950\ (M/M_S)\ m$$

Part B. Calculating the temperature of Hawking radiation from a black hole

Planck's constant, $h \approx 6.626\times10^{-34}$ J sec, governs the size of quantum effects. The Heisenberg uncertainty relation between measurements of momentum (Δp) and position (Δx) is:

$$(\Delta p)\ (\Delta x) \approx (h/4\pi)$$

8. Because photons may be emitted from any part of the event horizon, whose dimensions are described by the Schwarzschild radius R_s or diameter 2 R_s, there is an uncertainty $(\Delta x) \sim 2R_s$ for the initial position of the photons. What is the corresponding uncertainty in the momentum of emitted photons, Δp?

$$\Delta p \approx h/(8\ \pi\ R_s) = (h\ c^2)/(16\ \pi\ G\ M)$$

This uncertainty in momentum corresponds to uncertainty in the photon energy Δp c. That energy can be characterized in terms of thermal energy fluctuations $k_B T$ at a temperature T (with $k_B \approx 1.38\times10^{-23}$ Joules/Kelvin defined as Boltzmann's constant):

$$\Delta p\ c \approx k_B\ T$$

9. Equating these two expressions for Δp and solving for T, what is the temperature corresponding to Hawking radiation?

$$T \approx (h\ c^3)/(16\ \pi\ k_B\ G\ M)$$

10. Look up Stephen Hawking's derivation of the temperature of a black hole's event horizon. How close did your answer come to Hawking's?

Stephen Hawking's much more rigorous derivation of the temperature of a black hole's event horizon gave the answer:

$$T = (h\ c^3)/(16\ \pi^2\ k_B\ G\ M)$$

Our estimate was only off by a factor of π, which is not too shabby.

11. Plugging in the numbers for the constants, how would you express Hawking's value for the temperature of a black hole (in Kelvin) in terms of solar masses (relative to our sun)?

$$T \approx [(6.626\times10^{-34})(3.00\times10^{8})^3]/[(16\ \pi^2)(1.38\times10^{-23})(6.67\times10^{-11})(1.99\times10^{30})]\ (M_s/M)\ K$$

$$\approx (M_s/M)\ 6.18\times10^{-8}\ K$$

Note how cold this is! The surface of a black hole the mass of our sun is barely warm. Such a black hole barely emits particles. Much less massive black holes are hotter and emit more particles.

Part C. Advanced extension (using calculus): Calculating the time for a black hole to evaporate by emitting Hawking radiation

The Stefan-Boltzmann constant is:

$$\sigma_{SB} = \frac{(2\,\pi^5 * k_B^4)}{(15 * h^3 * c^2)} \approx 5.67\times10^{-8}\,\frac{\text{watt}}{m^2 * K^4}$$

A stationary black hole has an energy $E = M*c^2$. Due to Hawking radiation, a black hole loses energy at a rate (d/dt is the derivative with respect to time, or the change per time):

$$\left(\frac{d}{dt}\right) * (M * c^2) = -(4 * \pi * R^2)(\sigma_{SB} * T^4)$$

12. What is the equation if you plug in the expressions for the Schwarzschild radius, Hawking's temperature for the black hole and the equation (not the number) for the Stefan-Boltzmann constant?

$(d/dt)\,(M c^2) = -\,(h c^6)/\,(30720\,\pi^2\,G^2\,M^2)$

13. If you plug in numbers and express the black hole's mass in terms of solar masses, what is the power of Hawking radiation from a black hole?

$(d/dt)\,(M c^2) = -\,(M_s/M)^2\; 9.04\times10^{-29}\; W$

That is a tiny amount of radiation from a stellar-mass black hole and therefore a tiny energy loss from the total energy of the black hole. But black holes with smaller masses have higher temperatures and lose energy more rapidly than more massive black holes. Mini black holes should completely evaporate in one final explosion of Hawking radiation, unless other presently unknown quantum gravity effects intervene.

You can integrate your equation from question 12 above to find the time $t_{evaporation}$ (t_{evap}) for a black hole of initial mass M_0 to evaporate via Hawking radiation.

Below is all the calculus you need to know. Ignoring various multiplicative factors, your equation has the following starting point, which you can then separate out variables and integrate to solve:

$$\frac{dM}{dt} = -\frac{1}{M^2}$$

$$dt = -M^2 * dM$$

$$\int_0^{t_{evap}} dt = -\int_{M_0}^{0} M^2 * dM$$

$$\int_0^{t_{evap}} dt = \int_0^{M_0} M^2 * dM$$

$$t_{evap} = \frac{M_0^3}{3}$$

14. Use your equation from question 12 and the above calculus trick (but with all the relevant constants included this time) to find the time it takes for a black hole of initial mass M_0 to evaporate via Hawking radiation:

$$t_{evap} = (10240\, \pi^2\, G^2)(h\, c^4)\, M_0^3$$

15. Use your answer to question 14 and plug in the constants to find the evaporation time (in seconds or years) for a black hole with an initial mass expressed in multiples of solar masses or in kg:

$$t_{evap} \approx (M_0/M_s)^3\ 6.60\times10^{74}\ sec$$

$$\approx (M_0/M_s)^3\ 2.09\times10^{67}\ years$$

$$\approx (M_0/10^{11}\ kg)^3\ 10\ billion\ years$$

16. Have black holes evaporated since the Big Bang, which occurred 13.8 billion years ago? Do a sample calculation and explain your thought process.

Hawking theorized that mini black holes could have been formed soon after the Big Bang, when the universe was denser. Most stellar-mass black holes may have formed later, as large stars ultimately burned out one by one. Thus a small black hole of ~ 10^{11} kg or less created early in the 13.8-billion-year history of the universe would have evaporated by now. But more typical stellar-mass black holes would require far longer to evaporate than the universe's predicted remaining lifetime of about 5 billion years.